Bonus Guidelines. You can choose three exercises from the ten exercises below. Each correct solution earns you 1 point, while an incorrect one yields 0 points. Please note:

1. You're welcome to write your solutions in either German or English.
2. Minor computational errors are accepted, as long as they don't simplify the exercise.
3. **Submitting more than three exercises will raesult in disregarding ALL exercises (earning zero points).**
4. Use the "Bonus" option on the SAM-Up tool to submit your solutions. Ensure that **you are connected to an ETH WiFi or using a VPN**.
5. Make sure to upload your solutions before Wednesday 13.12. at 12:00.

The bonus added to your grade (before rounding) follows the formula:

Points	Bonus
1	0.1
2	0.2
3	0.25

Table 1: Conversion Table for Points to Bonus.

Exercise 1. Compute

$$a := \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}.$$

i.e. the limit of the recursive sequence $(a_n)_{n\in\mathbb{N}}$ with $a_0 = 1$ and $a_{n+1} = \sqrt{1+a_n}$.

Clarification: You also need to show that the sequence converges.

Solution. We first show that (a_n) is a bounded and increasing sequence. Then by Theorem 2.108 it converges and the limit must satisfy

$$a = \sqrt{1+a} \tag{1}$$

- (a_n) *is bounded:* We show

  $$\forall n \in \mathbb{N} : 1 \leq a_n \leq 2 \tag{2}$$

  by induction:

  - $a_0 = 1$. Clearly $1 \leq a_0 \leq 2$
  - Assume $1 \leq a_n \leq 2$. Then

    $$a_{n+1} = \sqrt{1+a_n} > \sqrt{1} = 1$$

    So the lower bound holds. Similarly,

    $$a_{n+1} = \sqrt{1+a_n} \leq \sqrt{1+2} \leq \sqrt{4} = 2.$$

- (a_n) *is increasing:* Again we use a proof by induction:
  - $a_1 = \sqrt{1+1} > \sqrt{1} = 1 = a_0$
  - Assume now $a_n > a_{n-1}$. Then we have

    $$\begin{aligned} a_n \geq a_{n-1} \quad &\iff \quad 1 + a_n \geq 1 + a_{n-1} \\ &\iff \quad \sqrt{1+a_n} \geq \sqrt{1+a_{n-1}} \\ &\iff \quad a_{n+1} \geq a_n \end{aligned}$$

So by Theorem 2.108 $\lim_{n\to\infty} a_n = a$ exists. Then

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1+a_n} = \sqrt{1 + \lim_{n\to\infty} a_n} = \sqrt{1+a},$$

where we used sequential continuity of the square root in the last step. We compute

$$a = \sqrt{1+a} \quad \Longrightarrow \quad a^2 = 1 + a$$
$$\Longrightarrow \quad a^2 - a - 1 = 0$$
$$\Longrightarrow \quad a_{1,2} = \frac{1 \pm \sqrt{1+4}}{2}$$

Notice that $\frac{1-\sqrt{5}}{2} < 0$ is not an option for the limit because $1 \leq a_n \leq \frac{1+\sqrt{5}}{2}$ for all n. So we conclude

$$a = \frac{1+\sqrt{5}}{2}.$$

Exercise 2. Show the equivalence of the following two statements:

i) Every Cauchy sequence in $\mathbb{R}$ converges.

ii) Every absolutely convergent series in $\mathbb{R}$ is convergent.

Clarification: It is not sufficient to say that i) was proved in the lecture. You need to show that i) implies ii) and that ii) implies i).

Solution.

$\Rightarrow$ Assume that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series and let $S_N = \sum_{n=1}^{N} a_n$ denote the partial sums. For $N > M$, we have

$$|S_N - S_M| = |\sum_{n=M+1}^{N} a_n| \leq \sum_{n=M+1}^{N} |a_n| \leq \sum_{n=M+1}^{\infty} |a_n|$$

and since the series is absolutely convergent, there is a $N_0 \in \mathbb{N}$ such that

$$\sum_{n=M+1}^{\infty} |a_n| \leq \epsilon$$

whenever $M \geq N_0$. Therefore, S_N is a Cauchy sequence and converges by i).

$\Leftarrow$ Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a Cauchy sequence. For all $i \in \mathbb{N}$, we can find a $N(i) \in \mathbb{N}$ such that

$$|a_n - a_m| < \frac{1}{2^i} \qquad \text{if} \qquad m, n \geq N(i).$$

Without loss of generality, we choose the $N(i)$ increasing i.e. such that $N(i+1) \geq N(i)$.

Then we consider the series

$$\sum_{i=1}^{\infty} |a_{N(i)} - a_{N(i+1)}|.$$

Note that since $|a_{N(i)} - a_{N(i+1)}| \leq \frac{1}{2^i}$, this series is bounded from above by the geometric series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. By ii) it follows that

$$\sum_{i=1}^{\infty} (a_{N(i)} - a_{N(i+1)})$$

converges.

However, this is a "telescopic series" and we get that

$$a_{N(k)} = a_{N(1)} + \sum_{i=1}^{k} (a_{N(i)} - a_{N(i+1)}).$$

It follows that $a_{N(k)}$ converges to some $a \in \mathbb{R}$. We conclude by Exercise 2.123.

Exercise 3. The exercises a) and b) are independent.

a) In this exercise, we want to show the following statement :

Let $a, b \in \mathbb{R}$ with $a < b$ and assume that $f : [a, b] \to \mathbb{R}$ is differentiable. Show that for every real number $y \in \mathbb{R}$ between $f'(a)$ and $f'(b)$, there is a $c \in [a, b]$ such that $f'(c) = y$.

Note that we only assume that f' exists, not that it is continuous. To prove this result, proceed as follows:

i) Assume that $f'(a) > 0$ and $f'(b) < 0$. Show that there exists a $c \in [a, b]$ such that $f'(c) = 0$.

Hint: Look for a Maximum.

ii) Assume that $f'(a) > f'(b)$ and show that for each real number $y \in (f'(b), f'(a))$, there is a $c \in [a, b]$ such that $f'(c) = y$.

iii) Show the statement.

b) i) Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable such that $f' > 0$. Show that f is injective.

ii) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose derivative is bounded on $\mathbb{R}$ by a constant $M > 0$ i.e. $|g'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that, for $a \in (-\frac{1}{M}, \frac{1}{M})$, the function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x + a \cdot g(x) \end{aligned}$$

is injective.

Solution.

a) i) Since f is differentiable, it is also continuous. Hence it assumes its maximum and its minimum on the compact interval $[a, b]$. The maximum is either at a, at b or at some point $c \in [a, b]$ where $f'(c) = 0$. We show that the first two cases are not possible. From this it follows that the third must hold and that we have found the required $c \in [a, b]$.

If a were a maximum of f, then for all $x \in [a, b]$ we had $f(x) \leq f(a)$. However, this would imply that

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \leq 0$$

contradicting our hypothesis that $f'(a) > 0$.

If b were a muximum of f, then for all $x \in [a, b]$ we had $f(x) \leq f(b)$. However, this would imply that

$$f'(b) = \lim_{h \to 0^+} \frac{f(b) - f(b-h)}{h} \geq 0$$

contradicting our hypothesis that $f'(b) < 0$.

ii) Consider the function

$$\begin{aligned} g : [a, b] &\to \mathbb{R} \\ x &\mapsto g(x) - yx. \end{aligned}$$

As it is the sum of two differentiable functions, g is differentiable and

$$g'(x) = f'(x) - y.$$

Since $f'(b) < y < f'(a)$ we have

$$g'(a) = f'(a) - y > 0 \qquad \text{and} \qquad g'(b) = f'(b) - y < 0.$$

Hence we can apply part i) to $g(x)$ and conclude that there is a $c \in [a, b]$ such that

$$g'(c) = f'(c) - y = 0.$$

iii) We have already addressed the case $f'(a) > f'(b)$.

If $f'(a) = f'(b)$, then the only value for y is $y = f'(a) = f'(b)$ and so we may choose either $c = a$ or $c = b$.

The case $f'(a) < f'(b)$ is analoguous to the case $f'(a) > f'(b)$ with the only modification that in part i), we have to consider f such that $f'(a) < 0$ and $f'(b) > 0$ and look for a minimum.

b) i) Assume by contradiction that f is not injective. Then there are points $x < y$ in I such that $f(x) = f(y)$. f is continuous and differentiable on $[x, y]$, and so by the mean value theorem there exists a $\alpha \in (x, y)$ such that

$$f'(\alpha) = \frac{f(x) - f(y)}{x - y} = 0.$$

This contradicts our assumption that $f' > 0$ and so the assumption that f was not injective must be false.

ii) Since the derivative of g is bounded by M, it holds for all $x \in \mathbb{R}$ that

$$|g'(x)| \leq M.$$

f is differentiable as a combination of differentiable functions and we have for all $x \in \mathbb{R}$

$$f'(x) = 1 + ag'(x).$$

We then see that

$$|f'(x) - 1| = |ag'(x)| \leq |a|M.$$

Hence we get that

$$1 - |a|M \leq f'(x) \leq 1 + |a|M.$$

But since $a \in (-\frac{1}{M}, \frac{1}{M})$, we have $|a| < \frac{1}{M}$ and so

$$1 - |a|M > 1 - 1 = 0$$

which immediately implies that $f' > 0$. We then conclude by part a) that f is injective.

Remark: This is an argument that you will see again in the course of Analysis II in the proof of the implicit function theorem.

Exercise 4.

a) Let I be a compact (i.e. closed and bounded) non-empty interval and $f\colon I \to \mathbb{R}$ a continuous function such that $f(I) \subseteq I$. Show that there is an $x \in I$ such that $f(x) = x$.

b) Show that statement a) is false if we assume that I is closed, but not necessarily bounded.

c) Show that statement a) is false if we assume that I is bounded, but not necessarily closed.

Solution.

a) Let $I = [a, b]$ with $a \leq b$ and consider the following continuous function

$$\begin{aligned} g : I &\to \mathbb{R} \\ x &\mapsto f(x) - x. \end{aligned}$$

Since $f(I) \subseteq I$, $a \leq f(x) \leq b$ applies to all $x \in [a, b]$. Hence g satisfies

$$g(a) = f(a) - a \geq 0 \qquad \text{and} \qquad g(b) = f(b) - b \leq 0.$$

According to the intermediate value theorem, there is therefore an $x \in [a, b]$ so that $g(x) = 0$ i.e. $f(x) = x$.

b) A counterexample is

$$\begin{aligned} f\colon [0, \infty) &\to [0, \infty) \\ x &\mapsto x + 1. \end{aligned}$$

The function f is continuous and maps $I = [0, \infty)$ to I but $f(x) - x = 1 \neq 0$ for all $x \in I$. Note that I is closed in $\mathbb{R}$ since its complement in $\mathbb{R}$ is the open set $(-\infty, 0)$.

c) A counterexample is the function

$$\begin{aligned} f : (-1, 1) &\to (-1, 1) \\ x &\mapsto \frac{x + 1}{2}. \end{aligned}$$

Clearly, $I = (-1, 1)$ is bounded but not closed. The function $f_{\text{ext}} : \mathbb{R} \to \mathbb{R}$ defined by $f_{\text{ext}}(x) = \frac{x+1}{2}$ has as only fixed point $x_0 = 1$ as

$$\frac{x + 1}{2} = x \qquad \Longleftrightarrow \qquad x = 1.$$

But $x_0 \notin I$.

Exercise 5.

a) Determine where the following functions $f : \mathbb{R} \to \mathbb{R}$ are discontinuous:

i) $f(x) = \frac{4x+5}{9-3x}$

ii) $f(x) = \frac{6}{x^2-3x-10}$

iii) $f(x) = \frac{9x^2+102x+289}{3x+17}$

iv)

$$f(x) = \begin{cases} 1-3x & x < -6 \\ 7 & x = -6 \\ x^3 & -6 < x < 1 \\ 1 & x = 1 \\ 2-x & x > 1. \end{cases}$$

v) $f(x) = \frac{1}{2-4\cos(\frac{x}{3})}$

Clarification: You also have to investigate what happens at the points where f might not be defined. Is there a continuous extension of the function?

b) Show that there exists at least one solution to the following equations in the indicated interval:

i) $w^2 - 4\log(5w+2) = 0$ on $[0,4]$,

ii) $4t + 10e^t - 2e^{2t} = 0$ on $[1,3]$.

c) Let $f_n : [0,1] \to \mathbb{R}$ be the sequence of functions given by

$$f_n(x) = \frac{n^2 x}{nx^2 + n^2 x + 1}.$$

Does this sequence of functions converge pointwise or uniformly? If possible, determine the limit.

Solution.

a) i) $x = 3$ since it is the unique zero of $g(x) = 9 - 3x$.

ii) There are two points where the function is not defined, namely at $x = -2$ and at $x = 5$. There are no possible continuous extensions since the limits from the right and from the left at these points are neither finite nor do they agree (one is actually $+\infty$ and the other is $-\infty$). Everywhere else the function is continuous as a combination of continuous functions.

iii) $9x^2 + 102x + 289 = (3x + 17)^2$ and so $f(x) = 3x + 17$ which is continuous everywhere.

iv) There is a jump at $x = -6$ since $(-6)^3 \neq 7$. At $x = 1$ the function is continuous.

v) Note that $2 - 4\cos(\frac{x}{3}) = 0$ if and only if $\cos(\frac{x}{3}) = \frac{1}{2}$. But $\cos(t) = \frac{1}{2}$ if $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for $k \in \mathbb{Z}$. So we get discontinuities at $x = \pi + 6k\pi$ and $x = 5\pi + 6k\pi$ for $k \in \mathbb{Z}$.

b) i) Note that the function $f(x) = x^2 - 4\log(5x + 2)$ is continuous as a combination/composition of continous function. Moreover, observe that $f(0) = -4\log(2) < 0$ and $f(4) = 16 - 4\log(22) > 0$. By the intermediate value theorem there is a $y \in [0, 4]$ such that $f(y) = 0$.

ii) Exactly the same argument as in i).

c) Note that $f_n(0) = 0$ for all $n \in \mathbb{N}$. For $0 < x \leq 1$ on the other hand, we have

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \frac{n^2x}{n^2x + nx^2 + 1} \\
&= \lim_{n\to\infty} \frac{x}{\frac{x^2}{n} + x + \frac{1}{n^2}} \\
&= \frac{x}{x} \\
&= 1.
\end{aligned}$$

Hence f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{else.} \end{cases}$$

The convergence can not be uniform: The functions f_n are continuous but f is clearly not. However, the uniform limit of any sequence of continuous functions is continuous. Hence f can not be a uniform limit of f_n.

Exercise 6.

a) Let $z \in \mathbb{C}$ be a complex number. Analyse the convergence behaviour of the series $\sum_{n=0}^{\infty} z^n$ and calculate the limit if it exists.

b) Let $\theta \in \mathbb{R}$. Analyse the convergence behaviour of the series $\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n}$ and calculate the limit if it exists.

Hint: You may use without proof that for $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ we have

$$\sum_{n=0}^{\infty} \mathrm{Re}\,(a_n) = \mathrm{Re}\left(\sum_{n=0}^{\infty} a_n\right).$$

Solution.

a) We assume first that $\sum_{n=0}^{\infty} z^n$ converges. By the same proof as Proposition 4.3, it follows that $\lim_{n\to\infty} z^n = 0$. But this implies that $\lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = 0$ by exercise 2.133. However, the sequence $(|z|^n)_{n\in\mathbb{N}}$ converges to 0 if and only if $|z| < 1$ so if $\sum_{n=0}^{\infty} z^n$ converges, we must have $|z| < 1$. In particular, for any z such that $|z| \geq 1$, the series $\sum_{n=0}^{\infty} z^n$ can not converge.

Reversely, assume now that $|z| < 1$. The series $\sum_{n=0}^{\infty} |z^n| = \sum_{n=0}^{\infty} |z|^n$ is a real geometric series which converges by the hypothesis $|z| < 1$.

We first compute the partial sums. By induction we see that

$$S_m = \sum_{n=0}^{m} z^n = \frac{1 - z^{m+1}}{1 - z}.$$

Indeed, for $m = 0$ we get 1 on both sides. Assume thus that the result holds for $m - 1$. Then

$$\sum_{n=0}^{m} z^m = \sum_{n=0}^{m-1} z^n + z^m = \frac{1 - z^m}{1 - z} + z^m = \frac{1 - z^m + (1 - z)z^m}{1 - z} = \frac{1 - z^{m+1}}{1 - z}.$$

Then we only need to compute the limit

$$\lim_{m\to\infty} S_m = \lim_{m\to\infty} \frac{1 - z^{m+1}}{1 - z} = \frac{1}{1 - z}.$$

b) *The series converges absolutely for any $\theta \in \mathbb{R}$.* This can be seen using the criterion of Majorant and the fact that $|\frac{\cos(n\theta)}{2^n}| \leq \frac{1}{2^n}$ and $\sum_{n=0}^{\infty} \frac{1}{2^n} < \infty$.

We then *compute the value of the series:*

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \sum_{n=0}^{\infty} \mathrm{Re}\left(\frac{e^{in\theta}}{2^n}\right)$$

$$\begin{aligned}
&= \operatorname{Re}\left(\sum_{n=0}^{\infty}\left(\frac{e^{i\theta}}{2}\right)^n\right) \\
&= \operatorname{Re}\left(\frac{1}{1-\frac{e^{i\theta}}{2}}\right) \\
&= \frac{1}{2-e^{i\theta}} + \frac{1}{2-e^{-i\theta}} \\
&= \frac{4-e^{i\theta}-e^{-i\theta}}{4-2(e^{i\theta}+e^{-i\theta})+e^{i\theta}e^{-i\theta}} \\
&= \frac{4-2\cos(\theta)}{5-4\cos(\theta)}.
\end{aligned}$$

Exercise 7.

a) Show that for any $\gamma > 0$ the "reciprocal" function $g : (\gamma, \infty) \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ is uniformly continuous.

b) Show that the function $h : \mathbb{R}_{>0} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ is not uniformly continuous on the non-negative reals.

Solution.

a) To show that $\forall \gamma > 0$ the function $g : (\gamma, \infty) \to \mathbb{R}, x \mapsto \frac{1}{x}$ is uniformly continuous we need to show that $\forall \gamma > 0$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in (\gamma, \infty)$ we have $|x-y| < \delta \Longrightarrow |g(x) - g(y)| < \varepsilon$.

We start by fixing any $\gamma > 0$ and taking an arbitrary $\varepsilon > 0$. We notice that for any $x, y \in (\gamma, \infty)$ we have that

$$|g(x) - g(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|x-y|}{xy} < \frac{|x-y|}{\gamma^2} \tag{3}$$

because $x, y > \gamma$. This allows us to deduce that for $\delta = \varepsilon\gamma^2$ we have

$$|x-y| < \delta \Longrightarrow |g(x) - g(y)| < \frac{|x-y|}{\gamma^2} < \frac{\delta}{\gamma^2} = \frac{\varepsilon\gamma^2}{\gamma^2} = \varepsilon \tag{4}$$

In fact, this is an even stronger result than uniform continuity because this shows that the reciprocal function on the domain (γ, ∞) for any $\gamma > 0$ is Lipschitz continuous with the Lipschitz constant γ^2.

b) To show that the function $h : \mathbb{R}_{>0} \to \mathbb{R}, x \mapsto \frac{1}{x}$ is **not** uniformly continuous we need to show that there exists an $\varepsilon > 0$ such that $\forall \delta > 0$ we can find $x, y \in \mathbb{R}_{>0}$ that satisfy $|x - y| < \delta \Longrightarrow |h(x) - h(y)| \geq \varepsilon$.

Heuristically, we expect that the outputs of two points that are close to zero (and each other) will still be far away from each other just because of the way the reciprocal function behaves. Therefore, it feels like the actual value of ε is not that important and we choose $\varepsilon = 1$. Then, we fix an arbitrary $\delta > 0$ and apply a corollary to the Archimedean principle (Corollary 2.65) to find an $n \in \mathbb{N}_{>0}$ such that $\frac{1}{n} < \delta$. This gives us

$$0 < \frac{1}{\delta} < n < n + 1 \leq 2n < 2n + 2,$$

which is equivalent to

$$\frac{1}{2n+2} < \frac{1}{2n} \leq \frac{1}{n+1} < \frac{1}{n} < \delta. \tag{5}$$

Now, if we choose our points x and y to be $x = \frac{1}{2n}$ and $y = \frac{1}{2n+2}$, we see that

$$|x - y| = \left|\frac{1}{2n} - \frac{1}{2n+2}\right| = \left|\frac{2}{4n^2 + 4n}\right| < \left|\frac{2}{4n}\right| = \frac{1}{2n} < \delta, \tag{6}$$

where we use (5) in the last inequality. This shows us that the distance between the two points x and y is less than δ. Furthermore, we have that

$$|h(x) - h(y)| = \left|\frac{1}{\frac{1}{2n}} - \frac{1}{\frac{1}{2n+2}}\right| = |2n - (2n + 2)| = 2 \geq 1 = \varepsilon. \tag{7}$$

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function.

a) Prove that the right and left derivative exist at every point.

b) Show that f is continuous.

Hint: Use part a).

Solution.

a) We consider $0 < h_1 < h_2$ and the corresponding points $x < x + h_1 < x + h_2$. By the definition of convexity in the form of Equation 5.7, we have

$$\frac{f(x + h_1) - f(x)}{h_1} \leq \frac{f(x + h_2) - f(x)}{h_2}$$

or in other words, the function

$$R(h) = \frac{f(x+h) - f(x)}{h}$$

is monotonically increasing on $(0, r)$ for any $r > 0$. It follows that

$$L(h) = R(-h) = \frac{f(x) - f(x-h)}{h}$$

is monotonically decreasing.

But then we see that looking at the triple $x - h < x < x + h$ as before, we obtain

$$\frac{f(x) - f(x-h)}{x - (x-h)} \leq \frac{f(x+h) - f(x)}{x + h - x}$$

which gives that $L(h) \leq R(h)$. In particular, $R(h)$ is bounded from below and hence its infimum $R_0 = \inf_{h \in (0,r)} R(h)$ is finite.

We now show that $\lim_{h \to 0^+} R(h) = R_0$, which shows that f is differentiable from the right at x. We pick an $\epsilon > 0$ arbitrary and look for a $\delta > 0$, such that for all $h \in [0, \delta)$ we have $R(h) \in (R_0 - \epsilon, R_0 + \epsilon)$. We always have $R(h) \geq R_0 - \epsilon$ by definition of R_0. $R_0 + \epsilon$ however is not a lower bound (it is strictly larger than the infimum) and so there is a $\delta \in (0, r)$ such that $R(\delta) \leq R_0 + \epsilon$. But since F is monotonically increasing, for each $h \in (0, \delta)$, we have $R(h) \leq R(\delta) < R_0 + \epsilon$ as wanted. Analoguously, we show that $\lim_{h \to 0^+} L(h)$ exists.

b) Remark 5.2 shows that differentiability from the right implies continuity from the right. Analoguously, one sees that differentiability from the left implies continuity from the left. But continuity from the left and from the right together imply continuity.

Exercise 9. Compute the value of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

proceeding as follows:

a) For every $n \in \mathbb{Z}$, compute the integral

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos(nx) - i \sin(nx)) \, dx$$

for $f(x) = x$.

b) Compute the value of the series using Parseval's identity (which you do not have to prove!)

$$|c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_{-n}|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.$$

Solution.

a) We compute the cases $n = 0$ and $n \neq 0$ separately:

$$\begin{aligned}
c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx \\
&= \frac{1}{2\pi} \frac{x^2}{2} \bigg|_{-\pi}^{\pi} \\
&= 0 \\
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} \, dx \\
&= \frac{1}{2\pi(-in)} \int_{-\pi}^{\pi} x(-in) e^{-inx} \, dx \\
&= \frac{1}{2\pi(-in)} \int_{-\pi}^{\pi} x \left(e^{-inx}\right)' dx \Big) \\
&= \frac{1}{2\pi(-in)} \left(x e^{-inx} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx} \, dx \right) \\
&= \frac{1}{2\pi(-in)} \left(\pi e^{-in\pi} - (-\pi e^{in\pi}) - \frac{e^{-inx}}{-in} \bigg|_{-\pi}^{\pi} \right) \\
&= \frac{1}{2\pi(-in)} \left(2\pi \cos(n\pi) - \left(\frac{e^{-in\pi}}{-in} - \frac{e^{in\pi}}{-in}\right) \right) \\
&= \frac{1}{2\pi(-in)} \left(2\pi(-1)^n + 2\frac{\sin(n\pi)}{n} \right) \\
&= \frac{(-1)^n}{-in} \\
&= \frac{i(-1)^n}{n}.
\end{aligned}$$

b) We first compute the left hand side of Parseval's identity

$$|c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_{-n}|^2 = 0 + 2\sum_{n=1} \frac{1}{n^2}$$

and conclude that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx\right)\\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}x^2\,dx\\ &= \frac{1}{4\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi}\\ &= \frac{1}{4\pi}\left(\frac{\pi^3}{3}-\frac{(-\pi)^3}{3}\right)\\ &= \frac{1}{4\pi}\frac{2\pi^3}{3}\\ &= \frac{\pi^2}{6}.\end{aligned}$$

Exercise 10. Consider the improper integral

$$\int_0^\infty \frac{\sin(x)}{x^\alpha}dx$$

with $\alpha > 0$. For which values of α does it converge?

Solution. We chop up the positive real line in pieces of length π and note that

$$\int_0^\infty \frac{\sin(x)}{x^\alpha}dx = \sum_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{\sin(x)}{x^\alpha}dx = \sum_{n=0}^{\infty}a_n$$

where we defined

$$a_n = \int_{n\pi}^{(n+1)\pi}\frac{\sin(x)}{x^\alpha}dx.$$

There are two possible sources of a divergence. Either one of the coefficients diverges or the series as a whole diverges. Since $\sin(x)/x^\alpha$ is finite everywhere except possibly at the origin, we actually only need to check the coefficient a_0.

For $\alpha \geq 2$ this coefficient does not exist: Since $\frac{\sin(x)}{x}$ is continuous (away from the origin) and $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, there is a $0 < r < \pi$ such that $\frac{\sin(x)}{x} \geq \frac{1}{2}$ for all $x \in [0, r]$. Hence

$$\int_0^r \frac{\sin(x)}{x^\alpha}dx \geq \frac{1}{2}\int_0^r \frac{1}{x^{\alpha-1}}dx.$$

But this integral converges if and only if $\alpha < 2$. Indeed, if $\alpha = 2$, then $\frac{1}{x^{\alpha-1}} = \frac{1}{x}$ integrates to the natural logarithm and $\lim_{x\to 0^+}\log(x) = -\infty$. If $\alpha > 2$, then $\frac{1}{x^{\alpha-1}}$ integrates to $\frac{x^{2-\alpha}}{2-\alpha}$ and again $\lim_{x\to 0^+}\frac{x^{2-\alpha}}{2-\alpha} = \infty$.

For $\alpha \in (0,2)$ the coefficient a_0 does exist. Indeed, the argument is almost as above. Since $\sin(x)/x \leq 1$, we see that

$$\int_0^\pi \frac{\sin(x)}{x^\alpha}dx \leq \int_0^\pi \frac{1}{x^{\alpha-1}}dx = \left.\frac{x^{2-\alpha}}{2-\alpha}\right|_0^\pi = \frac{\pi^{2-\alpha}}{2-\alpha}.$$

For $\alpha \in (0,2)$, the sequence a_n is an alternating sequence of decreasing size, so its series $\sum_n a_n$ converges. Moreover, a_n is alternating since $\sin(x+n\pi) = (-1)^n \sin(x)$. Finally, it is decreasing since $\frac{1}{x^\alpha}$ is monotonically decreasing.