Exercise 1. Practice further with integration using the Integral Trainer. The goal is for you to be able to calculate integrals, especially using partial integration and substitution, confidently and quickly.

Exercise 2. (Old exam question) Solve the initial value problem

$$\begin{cases} y' = xy^2 + x, \\ y(0) = 1. \end{cases}$$

Hint: First, read Remark 7.84 in the script.

Exercise 3. Solve the initial value problem

$$\begin{cases} y' - \left(\frac{4}{x} + 1\right) y = x^4, \\ y(1) = 1. \end{cases}$$

Exercise 4. Solve the following ordinary differential equations (ODEs):

a) $u''(x) + u(x) = \sin(2x), \qquad u(0) = 0, \quad u'(0) = 1.$
Hint: Look for a particular solution of the form $a\sin(2x) + b\cos(2x)$.

b) $u''(x) + 4u(x) = \cos(2x), \qquad u(0) = 1, \quad u'(0) = 0.$
Hint: Look for a particular solution of the form $ax\cos(2x) + bx\sin(2x)$.

c) $u''(x) + u'(x) - 2u(x) = x^2, \qquad u(0) = 2, \quad u'(0) = 1.$
Hint: Look for a particular solution of the form $ax^2 + bx + c$.

d) $u''(x) + 2u'(x) - 3u(x) = \cos(x) + x, \qquad u(0) = 1, \quad u'(0) = 1.$
Hint: Look for a particular solution of the form $a\sin(x) + b\cos(x) + cx + d$.

Exercise 5. (Old exam question) Consider the following differential equation for twice continuously differentiable functions $u : \mathbb{R}_{>0} \to \mathbb{R}$:

$$xu''(x) + 2u'(x) + \omega^2 xu(x) = 0,$$

where $\omega > 0$ is a fixed constant. Find all bounded solutions of this differential equation.

Hint: Consider the function $v(x) = x\,u(x)$.