Exercise 1. (Old exam question) Find all solutions $y : \mathbb{R} \to \mathbb{R}$ of the differential equation

$$y' = e^{x-y},$$

that are defined on the entire $\mathbb{R}$.

Exercise 2. (Old exam question) Find the solution $y\colon I \to \mathbb{R}$ of the initial value problem

$$x^2 y' = y^2, \quad y(1) = 2,$$

Exercise 3. (Old exam question) Find the general solution of the differential equation $y' = \frac{y+\sqrt{x^2+y^2}}{x}$ for $x > 0$.

Hint: Use the substitution $u = y/x$.

Differential Equations with Power Series In the following two Exercises, we want to learn a new technique for finding solutions to certain differential equations. The idea is to write the function as a Taylor series, differentiate term by term, substitute this into the differential equation, and derive conditions for the coefficients.

More specifically, we proceed as follows:

1. We assume that the solution $y(x)$ of a differential equation can be written as a Taylor series, that is
$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

2. We differentiate the Taylor series term by term and obtain
$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
and
$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

3. We substitute this into the differential equation and simplify (we may need to redefine the indices of the series).
4. We compare the coefficients with the same powers of x to determine the coefficients a_n.
5. We substitute the coefficients back into the Taylor series.

Exercise 4. Use the solution method with the Taylor series to find a solution to the differential equation

$$y'(x) + 2xy(x) = 0.$$

Do you know a function whose Taylor series is exactly the one you obtained as a solution? Verify that your solution is correct by substituting this function back into the differential equation.

Note: If you don't know such a function, solve the differential equation by separating variables and verify that the solutions match.

Exercise 5. (Bessel equation with $\alpha = 0$) Find a solution to the differential equation

$$x^2y'' + xy' + x^2y = 0.$$

Note: This is the DE from Example 7.73, item 4, for $\alpha = 0$.