Exercise 1. All elephants are pink. Proof by induction on the number of elephants: For 0 elephants, the assertion is correct. Let us inductively assume that in any set consisting of n elephants, all the elephants are pink. Let X be a set with $n+1$ elephants and we take out one elephant E. The remaining n elephants are all pink according to the induction assumption. We can choose E so that it is pink: If it would not be pink, we replace it with one of these n pink elephants. So all $n+1$ elephants are pink. Where is the mistake?

Exercise 2. Let $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Consider the equivalence relation $\sim_n$ on the integers $\mathbb{Z}$, which for $a, b \in \mathbb{Z}$ is given by

$$a \sim_n b \text{ exactly if } a - b \text{ is a multiple of } n.$$

Let C_n be the set of equivalence classes i.e. the quotient set.[1]

a) How many elements does C_n have?

b) For each number $a \in \mathbb{Z}$, let $[a]_n \in C_n$ be the associated equivalence class. Show that $(C_n, +)$ forms a group where the addition of two classes $[a]_n, [b]_n \in C_n$ is defined as
$$[a]_n + [b]_n := [a + b]_n.$$

c) Let $C_n^\times = C_n \setminus \{[0]_n\}$. Is the multiplication $\cdot$ inherited from the multiplication on $\mathbb{Z}$ i.e.
$$[a]_n \cdot [b]_n := [a \cdot b]_n$$
well-defined?

d) Does $(C_n^\times, \cdot)$ define a group? The existence of multiplicative inverses is not straightforward. Google for *extended Euclidean algorithm*.

e) Let $n, m \in \mathbb{N}$ and $k \in \mathbb{Z}$. Consider the proposition $g^k_{n,m} : C_n \to C_m$ given by $[a]_n \mapsto [k \cdot a]_m$. When does this rule define a well-defined mapping $g^k_{n,m}$?

Exercise 3. Try to practise your writing style. For example, to prove the rule $-0 = 0$, something like the following argument is expected:
The additive inverse -0 of 0 must satisfy the property $(-0) + 0 = 0 + (-0) = 0$ according to the definition of the inverse. But because the definition of the neutral element states that $0 + 0 = 0 + 0 = 0$ holds and because the inverse is unique in a group, $-0 = 0$ must hold.
Let $x, y, z \in \mathbb{R}$. Show the following further rules of calculation:

[1] See the explanations on the forum: `https://forum.math.ethz.ch/t/exercise-sheet-1/3011`.

1. $-(x+y) = (-x) + (-y)$ (where for the latter we also write $= -x - y$ write),
2. $-(x-y) = -x + y$,
3. Show that the distributive law for subtraction is. $x(y-z) = xy - xz$. holds.

Exercise 4. Let $(K, \leq)$ be an ordered field. Show that

$$x \leq y \iff x^3 + x \leq y^3 + y$$

holds for all $x, y \in K$.

Exercise 5. Let $(K, \leq)$ be an ordered field and let $x, y, z \in K$ with $xyz > 0$. Show that the following inequality holds.

$$\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Exercise 6. (*Inverse triangle inequality*) Let $(K, \leq)$ be an ordered field. Show that for all $x, y \in K$ the inequality.

$$||x| - |y|| \leq |x - y|$$

holds. In which cases does equality hold?

Exercise 7.

1. Let K be a field with finitely many elements, in short a finite field. Show that there is no order relation on K that would make K an ordered field.
2. Let K be a field. Suppose there is an element $u \in K$ with property $u^2 + 1 = 0$. Show that there is no order relation on K that would make K an ordered field.