**Exercise 1.** Let $X$ and $Y$ be sets, $\sim$ be an equivalence relation on $X$ and $\equiv$ be an equivalence relation on $Y$. Let $f : X \to Y$ be a function such that.

$$x_1 \sim x_2 \implies f(x_1) \equiv f(x_2)$$

holds for all $x_1, x_2$. Show that there exists a uniquely determined mapping $g : X/_\sim \to Y/_\equiv$ which is.

$$g([x]_\sim) = [f(x)]_\equiv$$

for all $x \in X$.

1. Suppose $f : X \to Y$ is surjective. Does it follow that $g$ is also surjective?
2. Suppose $f : X \to Y$ is injective. Does it follow that $g$ is also injective?
3. Suppose $g : X/_\sim \to Y/_\equiv$ is surjective. Does it follow that $f$ is also surjective?
4. Assume that $g : X/_\sim \to Y/_\equiv$ is injective. Does it follow that $f$ is also injective?

**Exercise 2.** Decide for yourself which of the following statements are true and which are false.
*Let $X$ and $Y$ be sets, and let $f : X \to Y$ be a function. Let $A$, $A_1$, $A_2$ be subsets of $X$, and $B$, $B_1$, $B_2$ be subsets of $Y$.*

(1) $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$
(2) $f(A_1) \cap f(A_2) = f(A_1 \cap A_2)$
(3) $f^{-1}(f(A)) = A$
(4) $f^{-1}(B_1) \cup f^{-1}(B_2) = f^{-1}(B_1 \cup B_2)$
(5) $f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2)$
(6) $f(f^{-1}(B)) = B$

Which of the false statements are true if we also assume that $f$ is an injective and a surjective function respectively?

**Exercise 3.** In this exercise we show the existence and uniqueness of a bijective function $\sqrt{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with property $(\sqrt{a})^2 = a$ for all $a \in \mathbb{R}_{\geq 0}$.

a) Show that for all $x, y \in \mathbb{R}_{\geq 0}$: $x < y$ is equivalent to $x^2 < y^2$.

b) *Uniqueness:* Derive from step 1 that for every $a \geq 0$ there can be at most one element $c \geq 0$ satisfying $c^2 = a$.

c) *Existence:* For a real number $a \in \mathbb{R}_{\geq 0}$ consider the non-empty subsets

$$X = \{x \in \mathbb{R}_{\geq 0} \mid x^2 \leq a\}, \qquad Y = \{y \in \mathbb{R}_{\geq 0} \mid y^2 \geq a\},$$

and apply the completeness axiom to find $c \in \mathbb{R}$ with $x \leq c \leq y$ for all $x \in X$ and $y \in Y$. Prove that $c \in X$ and $c \in Y$ to conclude that both $c^2 \leq a$ and $c^2 \geq a$ hold, so $c^2 = a$.

*Hint.* If by contradiction $c \notin X$ is (i.e. $c^2 > a$), then one can find a suitably small real number $\varepsilon > 0$ such that $(c-\varepsilon)^2 \geq a$. So $c - \varepsilon \in Y$, which contradicts $y \geq c$ for any $y \in Y$. The case of $c \notin Y$ is analogous.

We call *square root function* the function $\sqrt{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which assigns to each $a \in \mathbb{R}_{\geq 0}$ the number $c \in \mathbb{R}_{\geq 0}$ uniquely determined by the above construction. We note that $c^2 = a$, and we call $c = \sqrt{a}$ the *square root* of $a$. Show that:

d) *Increasing:* The function $\sqrt{\cdot}$ is increasing: for $x, y \in \mathbb{R}_{\geq 0}$ with $x < y$, the inequality $\sqrt{x} < \sqrt{y}$ holds.

e) *Bijectivity:* The function $\sqrt{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is bijective.

f) *Multiplicity:* For all $x, y \in \mathbb{R}_{\geq 0}$, $\sqrt{xy} = \sqrt{x}\sqrt{y}$.

g) *Two solutions:* Show that for $a > 0$ there are exactly two real solutions to the equation $x^2 = a$. How many are there for $a = 0$ and for $a < 0$?

**Exercise 4.** Which of the following subsets are open? Which are closed? Justify.

(a) The point $A = \{0\}$ in $\mathbb{R}$,

(b) The integers $\mathbb{Z}$ in $\mathbb{R}$,

(c) The interval $C = [0, \infty)$ in $\mathbb{R}$,

(d) The interval $D = (0, \infty)$ in $\mathbb{R}$,

(e) The set $E = \left\{ \frac{1}{n} \mid n \in \mathbb{N}, n > 0 \right\}$ in $\mathbb{R}$,

(f) The set $F = E \cup \{0\}$ in $\mathbb{R}$.

**Exercise 5.** Write the solutions of the following equations for $z \in \mathbb{C}$ in the form $z = a + bi$ with $a, b \in \mathbb{R}$.

| | | | | | |
|---|---|---|---|---|---|
| (a) | $z = (2+3i)(2+i)$ | (c) | $z = \frac{4+3i}{2-i}$ | (e) | $z^3 = i$ |
| (b) | $z = (2-i)^3$ | (d) | $z = \frac{2-i}{4+3i}$ | (f) | $z^2 + 3 + 4i = 0$ |