Exercise 1. Let $\mathbb{R}$ be a field of real numbers, and let $u \in \mathbb{R}$. Show that there is a unique element $c \in \mathbb{R}$, with $c^3 + c = u$. Accordingly, what can you say about the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + x$?

Hint 1: Use exercise 4 of Exercise Sheet 1 for the uniqueness.

Hint 2: Argue as in exercise 3c) of Exercise Sheet 2 for the existence.

Exercise 2. Find all complex numbers c with property $c^3 + c = 0$. Accordingly, what can you say about the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^3 + z$?

Compare with exercise 1.

Exercise 3. Draw the following subsets of the complex plane.

(a) $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 3\}$

(b) $\{z \in \mathbb{C} \mid \operatorname{Re}(z^2) \geq 0\}$

(c) $\{z \in \mathbb{C} \mid |z-1| \leq 3, \; |z| \leq 3, \; |z-i| \leq 3\}$

(d) $\{z \in \mathbb{C} \mid \operatorname{Re}(z+1) = \operatorname{Re}(z+iz), \; |z| \leq 2\}$

(e) $\{z \in \mathbb{C} \mid z \neq 1, \; |\frac{z}{z-1}| \leq 1\}$

Exercise 4. Find the infimum and the supremum of the following subsets of $\mathbb{R}$. Do they have a minimum, a maximum?

1. $A = \{a - b \mid a, b \in \mathbb{R}, 1 < a < 2, 3 < b < 4\}$,
2. $B = \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$,
3. $C = \{\frac{n}{m} \mid m, n \in \mathbb{N}, m + n \leq 10\}$.

Exercise 5. Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ be nonempty subsets bounded from above, with property that $x \geq 0$ for all $x \in X$ and $y \geq 0$ for all $y \in Y$. Show that

$$XY := \{xy \mid x \in X, y \in Y\}$$

is bounded from above, and that $\sup(XY) = \sup(X)\sup(Y)$ holds. What conventions for the symbol ∞ do you need if X or Y is unbounded?

Exercise 6. Let $X \subseteq \mathbb{R}$ be a nonempty subset with the following property: for all $x, y \in X$ and $c \in \mathbb{R}$, $x \leq c \leq y \implies c \in X$. Show that X is an interval.

Infer that any nonempty intersection of intervals is again an interval.

Exercise 7. Let $X \subset \mathbb{R}$ be a subset which is open and closed. Show that $X = \emptyset$ or $X = \mathbb{R}$ holds.

Hint: Assume X is nonempty, choose $x \in X$, and study the set of real numbers $r > 0$ with property $(x - r, x + r) \subseteq X$.