Exercise 1. Calculate the following limits in the real numbers if they exist:

$$\lim_{n\to\infty} \frac{7n^4+15}{3n^4+n^3+n-1}, \qquad \lim_{n\to\infty} \frac{n^2+5}{n^3+n+1}, \qquad \lim_{n\to\infty} \frac{n^5-10}{n^2+1}.$$

Do not use here or elsewhere any previously learned cooking recipe that you cannot justify.

Exercise 2. Formulate and prove a general theorem about limits of sequences as in Exercise 1.

Exercise 3. Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$, and let $F \subseteq \mathbb{R}$ be the set of accumulation points of the sequence $(x_n)_{n=0}^{\infty}$. Show that F is closed.

Exercise 4. Let $(x_n)_{n=0}^{\infty}$ be the sequence recursively defined by $x_0 = 1$ and

$$x_n = \frac{2}{3}\left(x_{n-1} + \frac{1}{x_{n-1}}\right)$$

for $n \geq 1$. Show that $(x_n)_{n=0}^{\infty}$ converges and determine the limit.

Exercise 5. Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ and $(c_n)_{n=0}^{\infty}$ be convergent sequences of real numbers, with limits A, B and C respectively. Let $(x_n)_{n=0}^{\infty}$ be the sequence defined by

$$x_n = \begin{cases} a_n & \text{if } n = 3k, k \in \mathbb{N} \\ b_n & \text{if } n = 3k+1, k \in \mathbb{N} \\ c_n & \text{if } n = 3k+2, k \in \mathbb{N}. \end{cases}$$

Calculate $\limsup_{n\to\infty} x_n$, $\liminf_{n\to\infty} x_n$ and the set of accumulation points of the sequence $(x_n)_{n=0}^{\infty}$.

Exercise 6. Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence of real numbers such that $(x_{n+1} - x_n)_{n=0}^{\infty}$ converges to 0. Set

$$A = \liminf_{n\to\infty} x_n \qquad \text{and} \qquad B = \limsup_{n\to\infty} x_n.$$

Show that the set of accumulation points of the sequence $(x_n)_{n=0}^{\infty}$ is the interval $[A, B]$. Construct an example of such a sequence with $[A, B] = [0, 1]$.

Exercise 7. Show that the limit superior does not commute with addition: Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be bounded sequences. Show that while

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,$$

holds, equality does not always hold.

Exercise 8. Let $(z_n)_{n=0}^{\infty}$ be a convergent sequence in $\mathbb{C}$. Show that $(|z_n|)_{n=0}^{\infty}$ converges and give the limit. Conversely, does the convergence of $(|z_n|)_{n=0}^{\infty}$ imply the convergence of $(z_n)_{n=0}^{\infty}$?