Exercise 1.

a) (Reading task) Read the text at the end of this exercise sheet. In particular, you should know how the factorial and the binomial coefficients are defined and know the binomial theorem and understand its proof.

   For a more detailed discussion of these topics, we recommend sections 3.3.1, 3.3.2 and 3.3.3 in the script by Prof. Manfred Einsiedler.

b) For any real number $a > 0$, we define the sequence of real numbers $(x_n)_{n=0}^{\infty}$ by $x_n = \sqrt[n]{a}$. Show that the sequence $(x_n)_{n=0}^{\infty}$ converges, and that.

$$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

c) We define a sequence of real numbers $(x_n)_{n=0}^{\infty}$ by $x_n = \sqrt[n]{n}$. Show that this sequence converges, with limit

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Exercise 2. Show that the "reciprocal" function $h : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $h(x) = \frac{1}{x}$ is continuous.

Deduce that functions $q : D \to \mathbb{R}$ of the type

$$x \mapsto q(x) = \tfrac{f(x)}{g(x)} \in \mathbb{R}$$

are continuous if $D \subseteq \mathbb{R}$ is a subset and $f : D \to \mathbb{R}$, $g : D \to \mathbb{R} \setminus \{0\}$ are continuous functions.

Exercise 3. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is not continuous.

Note: We will introduce the function sin properly later in the lecture. For this exercise, continuity of $\sin : \mathbb{R} \to \mathbb{R}$ and the usual properties you know from high school may be assumed.

Exercise 4. Show the following statements:

(a) Let $f : [0,1] \to [0,1]$ be a continuous mapping. Show that there exists an $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

(b) Let $g : [0,2] \to \mathbb{R}$ be a continuous mapping such that $g(0) = g(2)$ holds. Show that there exists an $x_0 \in [0,1]$ satisfying $g(x_0) = g(x_0 + 1)$.

Exercise 5. Let I be an interval and $f : I \to \mathbb{R}$ a continuous, injective mapping. Show that f is strictly monotone.

Exercise 6. Let I be an interval and $f : I \to \mathbb{R}$ a monotone mapping such that for all $a, b \in I$ and $\xi \in \mathbb{R}$ between $f(a)$ and $f(b)$ there exists a $x \in \mathbb{R}$ between a and b satisfying $f(x) = \xi$. Show that f is continuous.

Compare this result with the intermediate value theorem.

Exercise 7. Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ a function. Show that f is continuous if and only if for every open set $U \subseteq \mathbb{R}$ the preimage $f^{-1}(U)$ is also open.

Exercise 8. Which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are uniformly continuous? First convince yourself that each given function is continuous and sketch the graph.

a) $f(x) = \sqrt{|x|}$,

b) $f(x) = x^2$,

c) $f(x) = \min(\sqrt{|x|}, x^2)$,

d) $f(x) = \inf_{k \in \mathbb{Z}} |x - k|$,

e) $f(x) = \inf_{k \inf \mathbb{Z}} |x - k^2|$

f) $f(x) = x \cdot \inf_{k \in \mathbb{Z}} |x - k|$

Material for Exercise 1a)

Factorial The function $n \in \mathbb{N}_0 \mapsto n! \in \mathbb{N}$ is defined by

$$0! = 1,\ n! = \prod_{k=1}^{n} k.$$

The number $n!$ is called n-**factorial**.

Combinatorial meaning. There are exactly $n!$ different ways to sort the set $\{1, \ldots, n\}$ or also $n!$ possibilities for different orders when n numbered balls are drawn at random from an urn.

Binomial Coefficients For $n, k \in \mathbb{N}_0$ with $0 \leq k \leq n$, we define the **binomial coefficient** $\binom{n}{k}$, pronounced as "n over k", by

$$\binom{n}{k} = \frac{n!}{k!\ (n-k)!}.$$

If we replace k with n in the binomial coefficient by $n - k$, the two expressions in the denominator merely swap and we get for all $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$

$$\binom{n}{k} = \binom{n}{n-k}.$$

Addition formula For $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ with $1 \leq k \leq n$ hold $\binom{n}{0} = \binom{n}{n} = 1$ and

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}. \tag{1}$$

In particular, $\binom{n}{k} \in \mathbb{N}$ for all $n, k \in \mathbb{N}_0$ with $0 \leq k \leq n$.

Proof. We use the definition of the binomial coefficients and obtain

$$\binom{n}{0} = \binom{n}{n} = \frac{n!}{0!\ n!} = 1$$

and

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\ (n-(k-1))!} + \frac{n!}{k!\ (n-k)!}$$

$$
\begin{aligned}
&= \frac{k\ n!}{k!\ (n+1-k)!} + \frac{(n+1-k)\ n!}{k!\ (n+1-k)!} \\
&= \frac{(k+n+1-k)\ n!}{k!\ (n+1-k)!} \\
&= \binom{n+1}{k}
\end{aligned}
$$

by extension with k and $n+1-k$ respectively.

The statement that $\binom{n}{k} \in \mathbb{N}$ for all $k \in \mathbb{N}_0$ with $0 \leq k \leq n$, follows from the first two statements and induction on n. □

Combinatorial meaning The number $\binom{n}{k}$ for $n, k \in \mathbb{N}_0$ with $0 \leq k \leq n$ is the number of ways k elements can be selected from a collection with n elements. Formally speaking, there are exactly $\binom{n}{k}$ subsets of $\{1, \ldots, n\}$ that have k elements.

Binomial theorem For $w, z \in \mathbb{C}$ and $n \in \mathbb{N}_0$ holds.

$$
(w+z)^n = \sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^k.
$$

Proof of the binomial theorem. For $n = 0$ the proposition holds, since.

$$
(w+z)^0 = 1 = \sum_{k=0}^{0} 1 w^{0-k} z^k.
$$

Suppose the statement of the theorem holds for a $n \in \mathbb{N}_0$. Then we get

$$
\begin{aligned}
(w+z)^{n+1} &= (w+z)^n (w+z) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^k \right) (w+z) \\
&= \sum_{k=0}^{n} \binom{n}{k} w^{n+1-k} z^k + \sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^{k+1} \\
&= w^{n+1} + \sum_{k=1}^{n} \binom{n}{k} w^{n+1-k} z^k + \sum_{j=0}^{n-1} \binom{n}{j} w^{n-j} z^{j+1} + z^{n+1} \\
&= w^{n+1} + \sum_{k=1}^{n} \binom{n}{k} w^{n+1-k} z^k + \sum_{k=1}^{n} \binom{n}{k-1} w^{n+1-k} z^k + z^{n+1}
\end{aligned}
$$

$$
\begin{aligned}
&= w^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} w^{n+1-k} z^k + z^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} w^{n+1-k} z^k
\end{aligned}
$$

using an index shift and the addition formula. $\square$