Exercise 1. Let $D \subset \mathbb{R}$ be a subset. A function $f : D \to \mathbb{R}$ is called *Lipschitz-stetig* if there exists an $L \geq 0$ such that $|f(x) - f(y)| \leq L|x - y|$ holds for all $x, y \in D$.

a) Show that a Lipschitz continuous function is also uniformly continuous.

b) Show that the root function $[0, 1] \to \mathbb{R}$ given by $x \mapsto \sqrt{x}$ is uniformly continuous but not Lipschitz continuous.

c) Show that the root function $[1, \infty) \to \mathbb{R}$ is Lipschitz continuous and uniformly continuous.

Exercise 2. Let $a \in \mathbb{R}$. Calculate the following limits:

a) $\lim_{x\to 2} \frac{x^3 - x^2 - x - 2}{x - 2}$,

b) $\lim_{x\to\infty} \frac{3e^{2x} + e^x + 1}{2e^{2x} - 1}$,

c) $\lim_{x\to\infty} \frac{e^x}{x^a}$,

d) $\lim_{x\to\infty} \frac{\log(x)}{x^a}$.

In each case, choose a suitable domain on which the given formula defines a function.

Exercise 3.

a) Show the following asymptotics for $x \to \infty$:

i) $2x^3 + 3x^2 = O(x^3)$

ii) $x^p = O(\exp(x))$ for any natural number $p > 0$

iii) $\log(x) = O(x^{\frac{1}{p}})$ for any natural number $p > 0$

b) Show the following asymptotics for $x \to \infty$:

i) $x^p = o(x^q)$ for natural numbers $0 < p < q$

ii) $x^p = o(\exp(x))$ for all natural numbers $p > 0$

iii) $\log(x) = o(x^p)$ for all natural numbers $p > 0$

c) Show the following asymptotics for $x \to 0$:

i) $x^q = o(x^p)$ for all the natural numbers $0 < p < q$.

ii) $\exp(x) = 1 + o(1)$

Exercise 4. Find an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the limit of the sequence $(f(n))_{n=0}^{\infty}$ exists, but not the limit $\lim_{x\to\infty} f(x)$. Interpret this in terms of Lemma 3.70. and the following question: For a real number A, how does one translate the statement.

$$\lim_{x\to\infty} f(x) = A$$

correctly into a statement about convergence of sequences?

Hint: Exercise 8 of sheet 5 could provide inspiration for a counterexample.

Exercise 5. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers with $0 \leq a_n$ such that $\sum_{n=0}^{\infty} a_n$ converges. Show that then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ also converges.

Exercise 6. Show that the following series of real numbers converge and calculate their values. Check your result with Wolframalpha.

a) $\sum_{n=0}^{\infty} x^n$ for a real number $x \in \mathbb{R}$ with $|x|<1$,

b) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n-1}+1}{3^n}$

c) $\sum_{n=k}^{\infty} \frac{1}{5^n}$ for $k \in \mathbb{N}$,

d) $\sum_{n=0}^{\infty} \frac{n}{5^n}$,

e) $\sum_{n=0}^{\infty} \frac{n^2}{5^n}$.

Exercise 7. Let $s > 1$ be a real number. Use Proposition 4.14 to show that the series

$$\sum_{n=0}^{\infty} \frac{1}{n^s}$$

converges.