Exercise 1. (Old exam questions)

a) Show that $\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}$.

b) Does the following series converge? Does it converge absolutely?

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}.$$

Name the convergence rates that you use.

c) Calculate the radius of convergence of the series

$$x + 4x^4 + 9x^9 + 16x^{16} + \cdots = \sum_{n=0}^{\infty} n^2 x^{n^2}.$$

If you are using a result äbout convergence radii from the lecture, write down the complete statement of this result.

Exercise 2. Show that the following series converge or diverge. Calculate the limit if the series converges.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{\log(2^n)}{e^n}$$

Exercise 3. Let $(a_n)_{n=0}^{\infty}$ be a sequence of non-negative real numbers that converges to 0. We define

$$\prod_{n=0}^{\infty}(1+a_n) = \lim_{n\to\infty} \prod_{k=0}^{n}(1+a_k)$$

in the same sense as series. Show this:

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \prod_{n=0}^{\infty}(1+a_n) \text{ converges, with limit} \neq 0.$$

Hint: Prove and use that $\frac{1}{2}x \leq \log(1+x) \leq x$ holds for sufficiently small $x \in \mathbb{R}_{\geq 0}$.

Exercise 4. Decide which of the following series converge.

$$\text{a)} \sum_{n=0}^{\infty} \frac{n}{n^2+1} \quad \text{b)} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{c)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{d)} \sum_{n=0}^{\infty} \frac{1}{(n!)^2}$$

$$\text{e)} \sum_{n=1}^{\infty} \log(\tfrac{n+1}{n}) \quad \text{f)} \sum_{n=2}^{\infty} \frac{1}{\log(n)^n} \quad \text{g)} \sum_{n=2}^{\infty} \frac{1}{n(n+1)(n+2)} \quad \text{h)} \sum_{n=1}^{\infty} \log\left(1+\tfrac{1}{n^2}\right)$$

Exercise 5. (Hard)

a) Calculate the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{(\sqrt{n^2+n} - \sqrt{n^2+1})^n}{n^2} x^n.$$

b) Show convergence of the power series at the points $-R, R \in \mathbb{R}$.

Exercise 6. (Easy) Let $w = re^{i\theta}$ be non-zero. Show that the nth roots of w (namely the solutions $z \in \mathbb{C}$ of the equation $z^n = w$) are given by the n-numbers

$$\left\{ \sqrt[n]{r}\, e^{i\left(2\pi\alpha + \frac{\theta}{n}\right)} \,|\alpha = 0, \tfrac{1}{n}, \tfrac{2}{n}, \dots, \tfrac{n-1}{n} \right\}.$$

Note: If $w = 1$, its n-th roots are given by

$$\{e^{2\pi i\alpha} \,|\alpha = 0, \tfrac{1}{n}, \tfrac{2}{n}, \dots, \tfrac{n-1}{n}\}$$

and are called the n-th **roots of unity**.

Exercise 7. (Easy) For all natural numbers $n \geq 2$, show the identity $\sum_{k=0}^{n-1} e^{2\pi i \frac{k}{n}} = 0$. Illustrate the identity with a sketch for $n = 2, 3, 5$.