Exercise 1. Calculate the derivative of the following functions $f_i : \mathbb{R} \to \mathbb{R}$:

a) $f_1(x) = \sqrt{1+x^2}$

b) $f_2(x) = \cos(\cos x)$

c) $f_3(x) = \exp\left(\frac{1}{1+x^4}\right)$

d) $f_4(x) = \frac{e^x - 1}{e^x + 1}$

e) $f_5(x) = x^{\sin x}$ (defined only for $x > 0$)

Exercise 2. Let $D \subseteq \mathbb{R}$ be a non-empty subset without isolated points, and let $f, g \in C^n(D)$. Explain why $fg \in C^n(D)$ holds and show

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} .$$

Note: The aim of this exercise is to formulate the proof of Corollary 5.13.

Exercise 3. (Easy) Let a be any real number, and $f : \mathbb{R}_{>0} \to \mathbb{C}$ the function given by $f(x) = x^a$. Show that f is continuously differentiable and that $f'(x) = ax^{a-1}$.

Note: In this exercise we generalise Corollary 5.14.

Exercise 4. We consider the function

$$\begin{aligned} f_n : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \operatorname{sgn}(x) x^{n+1}. \end{aligned}$$

for every non-negative integer $n \in \mathbb{N}_0$.

a) Show that f_n is continuous.

b) Show that the derivatives $f_n', f_n^{(2)}, ..., f_n^{(n)}$ exist.

c) Calculate $f_n^{(n)}$ and conclude that the inclusions $C^{n+1}(\mathbb{R}) \subset C^n(\mathbb{R})$ are strict inclusions.

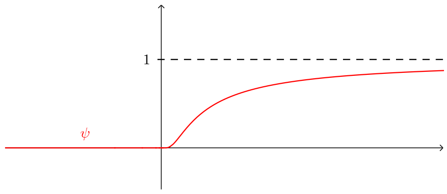

Figure 1: Graph of the function of exercise 5

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Exercise 5. (Hard) We consider the function $\psi : \mathbb{R} \to \mathbb{R}$, defined by

$$\psi : x \in \mathbb{R} \mapsto \begin{cases} \exp\left(-\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Show that ψ is smooth on $\mathbb{R}$ and that all its derivatives vanish at 0.

Hint 1: Show by induction that for $x > 0$

$$\psi^{(n)}(x) = \exp\left(-\frac{1}{x}\right) f_n\left(\frac{1}{x}\right)$$

where $f_n(x)$ is a polynomial.

Hint 2: To calculate the derivative at $x = 0$, use that the exponential function grows faster than any polynomial i.e.

$$\lim_{y \to \infty} \frac{y^n}{\exp(y)} = 0,$$

and set $x = \frac{1}{y}$.

Exercise 6. (Hard) For a fixed real number a, we consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} |x|^a \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Determine for which choice of $a \in \mathbb{R}$ the function f is continuous, differentiable or even continuously differentiable.