Exercise 1. (Warm up with MC questions from old exams) Exactly one answer is correct.

1. The function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \max(x, 1-x)$ is
   A) is not continuous.
   B) continuous but not differentiable.
   C) differentiable but not continuously differentiable.
   D) continuously differentiable but not twice differentiable.
2. Let $a < b$. What is true for all functions $f : [a, b] \to \mathbb{R}$?
   A) If f is continuous, then f is differentiable.
   B) If f is monotonically increasing, then f is bounded.
   C) If for every $x \in [f(a), f(b)]$ there is a $c \in [a, b]$ such that $f(c) = x$, then f is continuous.
   D) If f has a maximum and a minimum, then f is continuous.

Exercise 2. Calculate the following limits using l'Hôpital's rule.

(a) $\lim_{x \to 0} \frac{\sin(x) - x}{x^2 \sin(x)}$ (b) $\lim_{x \to 0} \frac{e^x - x - 1}{\cos x - 1}$ (c) $\lim_{x \to 2} \frac{x^4 - 4^x}{\sin \pi x}$

(d) $\lim_{x \to -\infty} x^3 e^x$ (e) $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}}$

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(x) = f'(x)$ holds for all $x \in \mathbb{R}$. Show that a real number c exists with $f(x) = c \cdot \exp(x)$.

Exercise 4. Show that there is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = \operatorname{sgn}(x)$ holds for all $x \in \mathbb{R}$.

Exercise 5. (Old Exam Exercise)

a) Let $I = (a, b)$ be an interval with $a < b$ in $\overline{\mathbb{R}}$ and $f : I \to \mathbb{R}$ a differentiable function. Show that if the derivative of f is bounded, then f is Lipschitz continuous, i.e. there exists a constant $c > 0$ with $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in I$.

b) Does the above statement also apply if the domain of f is not an interval? Prove or give a counterexample.

Exercise 6. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a convex function. Show that every local minimum of f is a global minimum.

Reminder: Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$. Then f is called *convex* if for all $a, b \in I$ with $a < b$ and all $t \in (0, 1)$, the inequality

$$f\big((1-t)a + tb\big) \leq (1-t)f(a) + tf(b)$$

is valid. We say that f is *strictly convex* if the inequality is strict.

Exercise 7. (Easy, part b is an old exam Exercise)

a) Show that $f : I \to \mathbb{R}$ is convex if and only if for all $x \in (a, b) \subset I$ the inequality

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x}$$

holds, and strictly convex if the above inequality is strict.

b) Let f now be a convex function $\mathbb{R} \to \mathbb{R}$ such that $f(x) \leq 2023$ for all $x \in \mathbb{R}$. Show that f is constant.

Exercise 8. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ a convex function, $x_1, x_2, \ldots, x_n \in I$ and $t_1, t_2, \ldots, t_n \in [0, 1]$ with $t_1 + t_2 + \cdots + t_n = 1$. Show that

$$f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \leq t_1f(x_1) + t_2f(x_2) + \cdots + t_nf(x_n)$$

is valid. This inequality is called *Jensen's inequality.*

Conclude from this that for any positive real numbers $x_1, x_2, \ldots x_n$ the inequalities

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leq \sqrt[n]{x_1x_2\cdots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}$$

apply.