You can take for granted the following facts:

- An algebra of sets on a set X is a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ that contains the set $\varnothing$ and is closed under complements and finite unions.
- A pre-measure λ on an algebra $\mathcal{A}$ is a map $\lambda : \mathcal{A} \to [0, +\infty]$ such that $\lambda(\varnothing) = 0$ and given pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{A}$ whose union $A = \bigcup_{k=1}^{\infty} A_k$ belongs to $\mathcal{A}$, then $\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$.

Exercise 1. (10 points)
Let $\mathcal{A}$ be an algebra of sets on a set X and $\lambda : \mathcal{A} \to [0, +\infty]$ a pre-measure on $\mathcal{A}$. We define the mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ as

$$\mu(E) := \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) \,\middle|\, A_1, A_2, \ldots \in \mathcal{A}, E \subseteq \bigcup_{k=1}^{\infty} A_k \right\}$$

for any set $E \subseteq X$. One can then show that μ is a measure.

(a) (2 points) State Carathéodory's measurability criterion with respect to μ.

(b) (4 points) Show that for every $A \in \mathcal{A}$, it holds that $\lambda(A) = \mu(A)$.

(c) (4 points) Prove that every set $A \in \mathcal{A}$ is μ-measurable.

Note: You do **not** need to show that μ is a measure but you **are allowed** to use that it satisfies the properties of a measure as long as you state them clearly.

Exercise 2. (14 points)

(a) (2 points) Let (Ω, μ) be a measure space. For $1 \leq p < +\infty$, define the space $L^p(\Omega, \mu)$.
Fix $1 \leq p < +\infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\Omega, \mu)$.

(b) (4 points) Show that there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \|f_{n_k} - f_{n_{k+1}}\|_{L^p(\Omega,\mu)} < +\infty.$$

(c) (4 points) Show that the function $g(x) := \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ is in $L^p(\Omega, \mu)$ and is finite μ-almost everywhere.

(d) (4 points) Prove that there exists a function $f \in L^p(\Omega, \mu)$ such that $\|f_{n_k} - f\|_{L^p(\Omega,\mu)} \to 0$ as $k \to \infty$ and deduce that $L^p(\Omega, \mu)$ is complete.

Exercise 3. (6 points)
Compute the following limits:

(a) (3 points)

$$\lim_{m\to\infty} \int_0^{\infty} \frac{1}{1 + x^m}\, dx$$

(b) (3 points)

$$\lim_{m\to\infty} \int_0^{1} \sum_{k=0}^{m} \frac{x^k}{k!}\, dx$$