Exercise 2.1.
Given a measure μ on a set X, we define the set of atoms of μ as

$$A_\mu := \{x \in X : \{x\} \text{ is measurable and } \mu(\{x\}) > 0\}.$$

(a) Assuming that $\mu(X) < +\infty$, show that A_μ is at most countable.

(b) Is the same true if μ is only assumed to be σ-finite? And in general? Show it or give a counterexample.

(c) ★ Construct an example of measure μ on an uncountable set X such that $\mu(\{x\}) > 0$ for every $x \in X$ but $\mu(X) < \infty$. This shows that the condition of the measurability of $\{x\}$ in the definition of A_μ cannot be removed.

Exercise 2.2. (Upper and lower semicontinuity of measures.)
Let $\mathcal{E}$ be a σ-algebra on a set X and $\mu : \mathcal{E} \to [0, \infty]$ a σ-additive function on $\mathcal{E}$. For a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$,
(a) show that

$$\mu\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mu(A_n).$$

(b) show that also

$$\limsup_{n\to\infty} \mu(A_n) \leq \mu\left(\limsup_{n\to\infty} A_n\right)$$

holds provided that $\mu(X) < \infty$.

Exercise 2.3. ♣
Let $\mathcal{E}$ be a σ-algebra on a set X and $\mu : \mathcal{E} \to [0, \infty]$ a σ-additive function on $\mathcal{E}$. Which of the following are true for an arbitrary sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$?
(a) Whenever $B \subseteq \bigcup_{n=1}^\infty A_n$,

$$\mu(B) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

(b)

$$\limsup_{n\to\infty} A_n^c = \left(\liminf_{n\to\infty} A_n\right)^c.$$

(c)

$$\mu\left(\liminf_{n\to\infty} A_n\right) = \liminf_{n\to\infty} \mu(A_n).$$

(d)

$$\limsup_{n\to\infty} \mu(A_n) \leq \mu\left(\limsup_{n\to\infty} A_n\right).$$

Exercise 2.4. ★

Let $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ be a measure with the following property: there is a real number $s > n$ such that for every $x \in \mathbb{R}^n$ and $r > 0$,

$$\mu(B(x, r)) \leq r^s.$$

Show that $\mu \equiv 0$. Here $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ denotes the open ball with center x and radius r in $\mathbb{R}^n$.