Exercise 3.1.
Denote by λ the Lebesgue measure on $\mathbb{R}$. Let $E \subset [0,1]$ be a Lebesgue measurable set of strictly positive measure, i.e. $\lambda(E) > 0$. Show that for any $0 \leq \delta \leq \lambda(E)$, there exists a measurable subset of E having measure exactly δ.
Hint: Consider the function $t \in [0,1] \mapsto \lambda([0,t] \cap E)$.

Exercise 3.2.
Let $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ be the function

$$\mu(A) := \sqrt{\mathcal{L}^1(A)}$$

for $A \subseteq \mathbb{R}$, where $\mathcal{L}^1$ denotes the Lebesgue measure.
(a) Show that μ is a measure.

(b) ★ What is the σ-algebra of μ-measurable sets?

Exercise 3.3.
Recall that the system of elementary sets is defined as

$$\mathcal{A} := \{A \subset \mathbb{R}^n \mid A \text{ is the union of finitely many disjoint intervals}\}.$$

(a) Prove that $\mathcal{A}$ is an algebra. To simplify the notation you may assume that $n = 1$.

(b) ★ Show that the volume function vol introduced in the lecture[1] for elementary sets is a pre-measure.

Remark: For $I = I_1 \times \ldots \times I_n$ an interval in $\mathbb{R}^n$, its volume is defined by

$$\text{vol}(I) = \prod_{k=1}^{n} \text{vol}(I_k),$$

where for an interval $I_k \subseteq \mathbb{R}$, $\text{vol}(I_k)$ is the length of I_k.

[1] Definition 1.3.1 in the Lecture Notes.

Exercise 3.4.
(a) Let X be any set with more than one element and consider the measure $\mu : \mathcal{P}(X) \to [0, +\infty]$ defined by:

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{else} \end{cases}.$$

Give an example of a non-μ-measurable subset.

(b) Define $\mathcal{A}$ to be the algebra in $\mathbb{R}$ generated by the half-closed intervals of the form $[a, b[$ for every $-\infty \leq a < b \leq \infty$. Note that any element in $\mathcal{A}$ can be expressed as the disjoint union of finitely many intervals of the type described before. Moreover, we define:

$$\lambda : \mathcal{A} \to [0, +\infty], \quad \lambda(A) = \begin{cases} +\infty & \text{if } A \neq \emptyset \\ 0 & \text{else} \end{cases}$$

Check that λ is a pre-measure and find two distinct Carathéodory-Hahn extensions of λ, i.e. two measures on $\mathcal{P}(\mathbb{R})$ which coincide with λ on $\mathcal{A}$. Why does this not yield a contradiction to the uniqueness statement Theorem 1.2.21 of the Lecture Notes?

Exercise 3.5. ♣
Let μ be a measure on $\mathbb{R}^n$ and $A, B_1, B_2, \ldots \subset \mathbb{R}^n$ be such that $A \subseteq \limsup_{k\to\infty} B_k$ and $\sum_{k=1}^{\infty} \mu(B_k) < \infty$. Which of the following statements are true?
(a) $\mu(A) > 0$.

(b) $\mu(A) = 0$.

(c) Every point of A belongs to infinitely many of the B_k.

(d) Every point of A belongs to all except possibly finitely many of the B_k.