Exercise 5.1.
Fix some $0 < \beta < 1/3$ and define $I_1 = [0,1]$. For every $n \geq 1$, let $I_{n+1} \subset I_n$ be the collection of intervals obtained removing from every interval in I_n its centered open subinterval of length β^n. Then define by $C_\beta = \bigcap_{n=1}^{\infty} I_n$, the *fat Cantor set* corresponding to β.
Show that:
(a) C_β is Lebesgue measurable with measure $\mathcal{L}^1(C_\beta) = 1 - \frac{\beta}{1-2\beta}$.

(b) C_β is not Jordan measurable. Indeed it holds $\underline{\mu}(C_\beta) = 0$ and $\overline{\mu}(C_\beta) = 1 - \frac{\beta}{1-2\beta} > 0$.

Exercise 5.2.
The goal of this exercise is to show that the Cantor triadic set C is uncountable. For that, recall quickly the construction of C: Every $x \in [0,1]$ can be expanded in base 3, i.e., can be written as $x = \sum_{i=1}^{\infty} d_i(x) 3^{-i}$ for $d_i(x) \in \{0,1,2\}$. The set C is then defined as the set of those $x \in [0,1]$ that do not have any digit 1 in their 3-expansion, i.e.:,

$$C := \{x \in [0,1] \mid d_i(x) \in \{0,2\}, \forall i \in \mathbb{N}\}.$$

Now, the Cantor-Lebesgue function F is defined by

$$F : C \to [0,1], \quad F\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) := \sum_{i=1}^{\infty} \frac{a_i}{2^{i+1}}.$$

(a) Show that $F(0) = 0$ and $F(1) = 1$.

(b) Show that F is well-defined and continuous on C.

(c) Show that F is surjective.

(d) Conclude that C is uncountable.

Exercise 5.3. ♣
Which of the following statements are true?
(a) There is a subset $A \subset \mathbb{R}$ which is not Lebesgue-measurable but such that the set $B := \{x \in A : x \text{ is irrational}\}$ is Lebesgue-measurable.

(b) There exist two disjoint sets $A, B \subset \mathbb{R}^n$ which are not $\mathcal{L}^n$-measurable but whose union is $\mathcal{L}^n$-measurable.

(c) If the boundary of $\Omega \subset \mathbb{R}^n$ has $\mathcal{L}^n$-measure zero, then Ω is $\mathcal{L}^n$-measurable.

(d) Let $A \subset [0,1]$ be a set which is not $\mathcal{L}^1$-measurable. Then the set $B := \{(x,x) : x \in A\} \subset \mathbb{R}^2$ is not $\mathcal{L}^2$-measurable.

Exercise 5.4.
In this exercise we want to prove that there is a one-to-one correspondence between the nondecreasing left-continuous[1] functions F on $\mathbb{R}$ with $F(0) = 0$ and the Borel measures on $\mathbb{R}$ that are finite on bounded Borel sets.
(a) Given any nondecreasing left-continuous function $F : \mathbb{R} \to \mathbb{R}$, show that the Lebesgue-Stieltjes measure Λ_F generated by F is the unique Borel measure on $\mathbb{R}$ that is equal to $F(b) - F(a)$ on $[a, b)$. Namely, for every other Borel measure μ on $\mathbb{R}$ such that $\mu([a, b)) = F(b) - F(a)$ we have that μ coincides with Λ_F on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

(b) Conversely, given any Borel measure μ on $\mathbb{R}$ that is finite on all bounded Borel sets, the function $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = \begin{cases} \mu([0, x)) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu([x, 0)) & \text{if } x < 0 \end{cases}$$

is nondecreasing and left-continuous and μ coincides with Λ_F on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

[1] A function $F : \mathbb{R} \to \mathbb{R}$ is left-continuous if $\lim_{x \to a^-} F(x) = F(a)$ for every $a \in \mathbb{R}$.