Exercise 6.1.
Let $f : [0,1] \to \mathbb{R}$ be an α-Hölder continuous function, namely, there is a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x-y|^\alpha$ for every $x, y \in [0,1]$, where $0 < \alpha \leq 1$ is a fixed number. Let $G = \{(x, f(x)) : x \in [0,1]\} \subset \mathbb{R}^2$ denote its graph.
(a) Show that $\mathcal{L}^2(G) = 0$.

(b) ★ Show moreover that $\mathcal{H}^s(G) = 0$ for every $s > 2 - \alpha$.

Exercise 6.2.
Show that the graph of the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

has Lebesgue measure zero in $\mathbb{R}^2$.

Exercise 6.3.
For $s \geq 0$ and $\emptyset \neq A \subset \mathbb{R}^n$, we define

$$\mathcal{H}^s_\infty(A) := \inf \left\{ \sum_{k \in I} r_k^s : A \subset \bigcup_{k \in I} B(x_k, r_k),\ r_k > 0 \right\},$$

where the set of indices I is at most countable. One can check that $\mathcal{H}^s_\infty$ is a measure. Prove that $\mathcal{H}^{1/2}_\infty$ is not Borel on $\mathbb{R}$.
Remark. Note that the definition of $\mathcal{H}^s_\infty$ coincides with Definition 1.8.1 in the Lecture Notes for $\delta = \infty$.

Exercise 6.4. ♣
Recall that a measure is called Radon if it is Borel regular and finite on compact sets.
Which of the following statements are true?
(a) The Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ is a Radon measure.

(b) For any nondecreasing and left-continuous function $F : \mathbb{R} \to \mathbb{R}$, the Lebesgue–Stieltjes measure Λ_F is a Radon measure on $\mathbb{R}$.

(c) For any $s > 0$, the Hausdorff measure $\mathcal{H}^s$ is a Radon measure on $\mathbb{R}^n$.

(d) The Dirac measure δ_0 is a Radon measure on $\mathbb{R}$.

(e) For every set $A \subset \mathbb{R}^n$, it holds $\mathcal{H}^{n+1}(A) = 0$.

Exercise 6.5.
Consider the continuous function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

and its graph

$$A := \{(x, f(x)) \mid x \in [0,1]\} \subset \mathbb{R}^2.$$

(a) ★ Show that $\mathcal{H}^1(A) = \infty$.

Hint: use the σ-additivity of $\mathcal{H}^1$ on Borel sets to decompose the curve A into pieces which "look like" straight lines and try to compare their $\mathcal{H}^1$-measure with their length.

(b) Show that $\mathcal{H}^s(A) = 0$ if $s > 1$.

(c) Conclude that $\dim_{\mathcal{H}}(A) = 1$.