

Exercise 7.1. ♣

Let μ be a measure on \mathbb{R}^n and $\Omega \subseteq \mathbb{R}^n$ a μ -measurable set. Which of the following statements are true?

(a) If $f : \Omega \rightarrow \mathbb{R}$ is μ -measurable and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function¹, then $g \circ f$ is μ -measurable.

(b) Let $f : [0, 1] \rightarrow \mathbb{R}$ and suppose that for every $c \in \mathbb{R}$, the set $\{x \in [0, 1] : f(x) = c\}$ is \mathcal{L}^1 -measurable. Then f is \mathcal{L}^1 -measurable.

(c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function which is continuous \mathcal{L}^1 -almost everywhere. Then f is \mathcal{L}^1 -measurable.

(d) Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of μ -measurable functions $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$. Then the set

$$E := \{x \in \mathbb{R}^n : \lim_{k \rightarrow \infty} f_k(x) \text{ exists and is finite}\}$$

is μ -measurable.

Exercise 7.2.

Let $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$. Show that the following statements are equivalent.

(i) $f^{-1}(U)$ is μ -measurable for every open set $U \subset \mathbb{R}$.

(ii) $f^{-1}(B)$ is μ -measurable for every Borel set $B \subset \mathbb{R}$.

(iii) $f^{-1}((-\infty, a))$ is μ -measurable for every $a \in \mathbb{R}$.

Exercise 7.3.

Let (X, μ, Σ) be a measure space and $f, g : X \rightarrow \mathbb{R}$ two measurable functions on X . Show that the sets $\{x \mid f(x) = g(x)\}$ and $\{x \mid f(x) < g(x)\}$ are measurable.

¹A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is Borel iff $g^{-1}(U)$ is a Borel set for every open set $U \subseteq \mathbb{R}$.

Exercise 7.4. ★

In this exercise, we construct a set which is Lebesgue measurable but not Borel, and use the construction to give an example of a continuous $G : \mathbb{R} \rightarrow \mathbb{R}$ and a Lebesgue measurable function $H : \mathbb{R} \rightarrow \mathbb{R}$ such that $H \circ G$ is not Lebesgue measurable.

(a) Let $h : [0, 1] \rightarrow [0, 1]$ be the Cantor function, which is the unique monotonically increasing extension of the function $F : C \rightarrow [0, 1]$ seen in Exercise 5.2, where $C \subset [0, 1]$ is the Cantor set. Define $g : [0, 1] \rightarrow [0, 2]$ by $g(x) := h(x) + x$. Show that g is strictly monotone and a homeomorphism.

(b) Show that $\mathcal{L}^1(g(C)) = 1$.

Hint: Use the natural decomposition of $[0, 1] \setminus C$ to deduce the result.

(c) Use Exercise 4.3 (a) to find a non-measurable subset $E \subset g(C)$ and define $A := g^{-1}(E)$. Show that A is a Lebesgue zero set and thus Lebesgue measurable.

(d) Show that A is not a Borel set.

Hint: Otherwise, the preimage of A with respect to continuous maps would necessarily be Borel and thus Lebesgue measurable as well.

(e) Find appropriate H, G as outlined above such that $H \circ G$ is not Lebesgue measurable, using the sets and functions introduced in the previous subtasks.

Exercise 7.5.

Let μ be a Borel measure on \mathbb{R} . Show that every monotone function $f : [a, b] \rightarrow \mathbb{R}$ is μ -measurable.