

Exercise 8.1.

Let μ be a measure on \mathbb{R}^n , $\Omega \subseteq \mathbb{R}^n$ a μ -measurable set and $f : \Omega \rightarrow [0, \infty]$ a μ -measurable function. Consider the sets $A_j \subseteq \Omega$ from Theorem 2.2.6 of the Lecture Notes, defined so that the sequence of functions

$$f_k = \sum_{j=1}^k \frac{1}{j} \chi_{A_j}$$

converges pointwise to f . Show that if f is bounded, then f_k converge uniformly to f , that is,

$$\sup_{x \in \Omega} |f(x) - f_k(x)| \longrightarrow 0 \text{ as } k \rightarrow \infty.$$

Exercise 8.2.

Let $f : \Omega \rightarrow \overline{\mathbb{R}}$ be μ -measurable. Prove that there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of simple functions $f_k : \Omega \rightarrow \mathbb{R}$ satisfying $|f_k(x)| \leq |f_{k+1}(x)|$ and $\lim_{k \rightarrow \infty} f_k(x) = f(x)$ for every $x \in \Omega$.

In particular, this shows that $|f_k(x)| \leq |f(x)|$ for each $x \in \Omega$ and $k \geq 1$.

Exercise 8.3. ♣

Which of the following assertions are true?

(a) Given a sequence $\{f_k\}$ of Lebesgue-measurable functions on \mathbb{R} converging to 0 in measure, there is a subset $A \subset \mathbb{R}$ of positive measure and a subsequence which converges uniformly on A .

(b) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, then f' is Lebesgue-measurable.

(c) There exists a function $f : \Omega \rightarrow [0, \infty)$ such that f is not measurable but \sqrt{f} is measurable.

(d) The sequence $f_n(x) = e^{-n(1-\sin x)}$ converges in measure to the function $f \equiv 0$ on any bounded interval $[a, b] \subset \mathbb{R}$.

Exercise 8.4.

Let $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ be \mathcal{L}^n -measurable functions, for $k \in \mathbb{N}$. Assume that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \mid |f_k(x) - f_{k+1}(x)| > 2^{-k}\}) < 2^{-k}$$

for all $k \in \mathbb{N}$. Show that the limit $\lim_{k \rightarrow \infty} f_k(x)$ exists almost everywhere.

Exercise 8.5.

Let μ be a measure on \mathbb{R}^n and $\Omega \subset \mathbb{R}^n$ be μ -measurable. Let $f: \Omega \rightarrow \overline{\mathbb{R}}$ be a finite, μ -measurable function, and $(f_k)_{k \in \mathbb{N}}$ a sequence of μ -measurable functions $f_k: \Omega \rightarrow \overline{\mathbb{R}}$.

(a) Suppose that every subsequence $(f_{k_j})_{j \in \mathbb{N}}$ contains a subsequence that converges to f in measure. Show that the whole sequence $(f_k)_{k \in \mathbb{N}}$ converges to f in measure.

(b) Show that the analogous statement from (a) is not true if we replace “convergence in measure” by “convergence pointwise almost everywhere”. Namely, show that there exists a sequence (f_k) and a function f such that every subsequence of (f_k) has a further subsequence that converges a.e. to f , but the whole (f_k) does not converge a.e. to any function.

Exercise 8.6. ★

Counterexample to $\varepsilon = 0$ in Lusin's Theorem: Find an example of a \mathcal{L}^1 -measurable function $f: [0, 1] \rightarrow \mathbb{R}$ such that for every \mathcal{L}^1 -measurable set $M \subset [0, 1]$ with $\mathcal{L}^1(M) = 1$, the restriction $f|_M: M \rightarrow \mathbb{R}$ is discontinuous in all but finitely many points of M .

Hint: You may use that there exists a Lebesgue measurable subset $A \subset [0, 1]$ such that

$$\mathcal{L}^1(U \cap A) \cdot \mathcal{L}^1(U \cap A^c) > 0$$

for all nonempty open subsets $U \subset [0, 1]$. Such a set A can be constructed using the fat Cantor set (see Exercise 1.6.2 in the lecture notes).

Exercise 8.7.

Counterexample to $\delta = 0$ in Egoroff's Theorem: Find an example of a sequence of \mathcal{L}^1 -measurable functions $f_k: [0, 1] \rightarrow \overline{\mathbb{R}}$ that converges pointwise almost everywhere to a \mathcal{L}^1 -measurable (\mathcal{L}^1 -almost everywhere finite) function $f: [0, 1] \rightarrow \overline{\mathbb{R}}$, but for every set $F \subseteq [0, 1]$ with $\mathcal{L}^1(F) = \mathcal{L}^1([0, 1])$ the convergence on F is not uniform.