Exercise 9.1. ♣
Which of the following statements are true?
4 correct answers are enough for the bonus.
(a) Let $\{f_k\}$ be a sequence of non-negative measurable functions on $\mathbb{R}$ such that $f_k \to f$ almost everywhere. Then $\lim_{k\to\infty} \int_{\mathbb{R}} f_k \, d\mathcal{L}^1$ exists and

$$\int_{\mathbb{R}} f \, d\mathcal{L}^1 \leq \lim_{k\to\infty} \int_{\mathbb{R}} f_k \, d\mathcal{L}^1.$$

(b) Let $f : [0,1] \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Then for each nonnegative integer k, $x^k f(x)$ is $\mathcal{L}^1$-summable in $[0,1]$.

(c) Let $f : (0,+\infty) \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Then $\lim_{x\to+\infty} |f(x)| = 0$.

(d) Let $f : (0,+\infty) \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Then there exists a sequence $x_n \to \infty$ such that $\lim_{n\to\infty} x_n f(x_n) = 0$.

(e) There exists a sequence $\{f_n\}$ of $\mathcal{L}^1$-summable functions on $(0,\infty)$ such that $|f_n(x)| \leq 1$ for all x and all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and $\lim_{n\to\infty} \int_{(0,\infty)} f_n \, d\mathcal{L}^1 = 1$.

(f) There exists a sequence $\{f_n\}$ of $\mathcal{L}^1$-integrable functions on $[0,1]$ such that $f_n \to 0$ pointwise and yet $\int_{[0,1]} f_n \, d\mathcal{L}^1 \to +\infty$.

Exercise 9.2.
(a) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of μ-measurable functions on a μ-measurable set $\Omega \subset \mathbb{R}^n$. Show that the series $\sum_{k=1}^{\infty} f_k(x)$ converges μ-almost everywhere, if

$$\sum_{k=1}^{\infty} \int_{\Omega} |f_k| d\mu < \infty.$$

(b) Let $\{r_k\}_{k\in\mathbb{N}}$ be an ordering of $\mathbb{Q}\cap[0,1]$ and $(a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be such that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. Show that $\sum_{k=1}^{\infty} a_k |x - r_k|^{-1/2}$ is absolutely convergent for almost every $x \in [0,1]$ (with respect to the Lebesgue measure).

Exercise 9.3.
Find an example of a continuous bounded function $f\colon [0,\infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$ and

$$\int_0^{\infty} |f(x)|^p dx = \infty \ ,$$

for all $p > 0$.

Exercise 9.4.
Let $f, g : \Omega \to \overline{\mathbb{R}}$ be μ-summable functions and assume that

$$\int_A f d\mu \leq \int_A g d\mu$$

for all μ-measurable subsets $A \subset \Omega$. Show that $f \leq g$ μ-almost everywhere. Moreover, conclude that, if

$$\int_A f d\mu = \int_A g d\mu$$

for all μ-measurable subsets $A \subset \Omega$, then $f = g$ μ-almost everywhere.

Exercise 9.5.
Let $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ be Lebesgue measurable functions. Find examples for the following statements.
(a) $f_n \to 0$ uniformly, but not $\int |f_n| dx \to 0$.

(b) $f_n \to 0$ pointwise and in measure, but neither $f_n \to 0$ uniformly nor $\int |f_n| dx \to 0$.

(c) $f_n \to 0$ pointwise, but not in measure.

Exercise 9.6.
Let $f : [0,1] \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Show that for a set $E \subset [0,1]$ of positive measure it holds that

$$f(x) \leq \int_{[0,1]} f(y)\, d\mathcal{L}^1(y)$$

for every $x \in E$.