Exercise 10.1. ♣
Which of the following statements are true?
(a) Let $\{f_k\}$ be a sequence of nonnegative $\mathcal{L}^1$-measurable functions on $\mathbb{R}$ converging uniformly to a function f. Then $\lim_{k\to\infty}\int_{\mathbb{R}} f_k \, d\mathcal{L}^1$ exists and

$$\int_{\mathbb{R}} f \, d\mathcal{L}^1 \leq \lim_{k\to\infty} \int_{\mathbb{R}} f_k \, d\mathcal{L}^1.$$

(b) Let $f_k : [0,1] \to [0,1]$ be $\mathcal{L}^1$-measurable functions for $k = 1, 2, \ldots$ and suppose that $f_k \to f$ almost everywhere. Then $\lim_{k\to\infty}\int_{[0,1]} f_k \, d\mathcal{L}^1$ exists and

$$\int_{[0,1]} f \, d\mathcal{L}^1 \leq \lim_{k\to\infty} \int_{[0,1]} f_k \, d\mathcal{L}^1.$$

(c) Let f be $\mathcal{L}^1$-summable on $\mathbb{R}$ and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ be $\mathcal{L}^1$-measurable subsets of $\mathbb{R}$. Then the limit $\lim_{n\to\infty}\int_{E_n} f \, d\mathcal{L}^1$ exists.

(d) Let $\{f_n\}$ be a sequence of continuous Lebesgue-summable functions on $[0,\infty)$ which converges to a Lebesgue-summable function f. Then

$$\lim_{n\to\infty} \int_{[0,\infty)} |f_n(x) - f(x)| \, \mathcal{L}^1(x) = 0.$$

Exercise 10.2.
Let $f : \mathbb{R} \to [0, +\infty]$ be $\mathcal{L}^1$-measruable. Assume that for all $n \geq 1$,

$$\int_{\mathbb{R}} \frac{n^2}{n^2 + x^2} |f(x)| \, d\mathcal{L}^1(x) \leq 1.$$

Show that

$$\int_{\mathbb{R}} |f| \, d\mathcal{L}^1 \leq 1.$$

Exercise 10.3.
Compute the limit

$$\lim_{n\to\infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x} \, dx.$$

Exercise 10.4. ★
Let f_k, f be $\mathcal{L}^1$-summable functions on $\mathbb{R}$ which are nonnegative $\mathcal{L}^1$-almost everywhere and satisfy the following additional hypotheses:

- $\liminf_{k\to\infty} f_k(x) \geq f(x)$ for $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$.
- $\limsup_{k\to\infty} \int_{\mathbb{R}} f_k(x)\, dx \leq \int_{\mathbb{R}} f(x)\, dx$.

Show that

$$\lim_{k\to\infty} \int_{\mathbb{R}} |f_k(x) - f(x)|\, dx = 0.$$

Exercise 10.5. ★
Let $0 < m < M < \infty$ be two real numbers and let $f : [0,1] \to \mathbb{R}$ be a measurable function satisfying $m \leq f(x) \leq M$ for almost every $x \in [0,1]$. Show that

$$\left(\int_{[0,1]} f(x)\, dx\right)\left(\int_{[0,1]} \frac{1}{f(x)}\, dx\right) \leq \frac{(m+M)^2}{4mM}$$

and characterize all functions for which equality holds.

Exercise 10.6.
For all $n \in \mathbb{N}$, let $f_n \colon [0,1] \to \mathbb{R}$ be defined by:

$$f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2}.$$

Prove that:
(a) $f_n(x) \leq \frac{1}{\sqrt{x}}$ on $(0,1]$ for all $n \geq 1$.

(b) $\lim_{n\to\infty} \int_0^1 f_n(x) dx = 0.$