Exercise 11.1. ♣

(a) Let $\{f_k\}$ be a sequence of $\mathcal{L}^1$-measurable functions on $[0,1]$ converging a.e. to a function f and such that $|f_k| \leq 100$ a.e. for each k. Is it true that

$$\lim_{k\to\infty} \int_{[0,1]} |f_k - f| \, dx = 0?$$

(b) Compute the limit

$$\lim_{k\to\infty} k \int_0^\infty e^{-kx} \sqrt{|\cos(x)|} \, dx.$$

(c) What is the value of the limit

$$\lim_{k\to\infty} \int_0^\infty e^{-x^k} \, dx?$$

(A) 0. (B) 1. (C) ∞. (D) None of the previous answers is correct.

(d) Let $f : \Omega \to [0,1]$ be a μ-measurable function with $\int_\Omega f \, d\mu > 0$. Is it true that

$$\lim_{k\to\infty} \int_\Omega f^{1/k} \, d\mu > 0?$$

Exercise 11.2.

Compute the limit

$$\lim_{n\to\infty} \int_a^{+\infty} \frac{n}{1+n^2x^2} \, dx$$

for every $a \in \mathbb{R}$.

Hint: recall that $\arctan x$ is a primitive of $\frac{1}{1+x^2}$.

Exercise 11.3.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ be μ-measurable and $f\colon \Omega \to [0,+\infty]$ be μ-summable. For all μ-measurable subsets $A \subset \Omega$ define (see Section 3.5 in the Lecture Notes)

$$\nu(A) = \int_A f d\mu.$$

(a) Prove that ν is a pre-measure on the σ-algebra of μ-measurable sets, hence we can define its Carathéodory-Hahn extension $\nu\colon \mathcal{P}(\Omega) \to [0,+\infty]$.

(b) Show that ν is a Radon measure.

(c) Prove that $\Sigma_\nu \supseteq \Sigma_\mu$ and that ν is absolutely continuous with respect to μ, that is, if $\mu(A) = 0$ then $\nu(A) = 0$.

Exercise 11.4.
Prove the following assertions.
(a) Let $f\colon [a,+\infty) \to \mathbb{R}$ be a locally bounded function and locally Riemann integrable. Then f is $\mathcal{L}^1$-summable if and only if f is absolutely Riemann integrable in the generalized sense (namely $\mathcal{R}\int_a^\infty |f(x)|dx = \lim_{j\to\infty} \mathcal{R}\int_a^j |f(x)|dx$ exists and it is finite) and in this case

$$\int_{[a,+\infty)} f(x) d\mathcal{L}^1 = \mathcal{R}\int_a^\infty f(x)dx = \lim_{j\to+\infty} \mathcal{R}\int_a^j f(x)dx.$$

(b) Let $f\colon [0,+\infty) \to \mathbb{R}$ be the function $f(x) = \frac{\sin x}{x}$, which is locally bounded and locally Riemann integrable. Show that f is Riemann integrable, i.e. $\mathcal{R}\int_0^\infty f(x)dx < +\infty$ but not absolutely Riemann integrable, i.e. $\mathcal{R}\int_0^\infty |f(x)|dx = \infty$. Hence f is not $\mathcal{L}^1$-summable.

Exercise 11.5.
This exercise is a more general version of Theorem 3.4.1 from the lecture notes.
(a) Let μ be a Radon measure on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a μ-measurable subset. Consider a function $f\colon \Omega \times (a,b) \to \mathbb{R}$, for some interval $(a,b) \subset \mathbb{R}$, such that:

- the map $x \mapsto f(x,y)$ is μ-summable for all $y \in (a,b)$;
- the map $y \mapsto f(x,y)$ is differentiable in (a,b) for every $x \in \Omega$;
- there is a μ-summable function $g\colon \Omega \to [0,\infty]$ such that $\sup_{a<y<b} |\frac{\partial f}{\partial y}(x,y)| \leq g(x)$ for all $x \in \Omega$.

Then $y \mapsto \int_\Omega f(x,y) d\mu(x)$ is differentiable in (a,b) with

$$\frac{d}{dy}\left(\int_\Omega f(x,y) d\mu(x)\right) = \int_\Omega \frac{\partial f}{\partial y}(x,y) d\mu(x)$$

for all $y \in (a,b)$.

(b) ★ Compute the integral

$$\phi(y) := \int_{(0,\infty)} e^{-x^2 - y^2/x^2} d\mathcal{L}^1(x)$$

for all $y > 0$.

Hint: use part (a) to obtain that ϕ solves the Cauchy problem

$$\begin{cases} \phi'(y) = -2\phi(y) & \text{for } y > 0 \\ \lim_{y\to 0^+} \phi(y) = \sqrt{\pi}/2. \end{cases}$$

Exercise 11.6.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$ and $f, f_k : \Omega \to \overline{\mathbb{R}}$ μ-summable functions.
(a) Show that Vitali's Theorem implies Dominated Convergence Theorem.

(b) Let $\Omega = [0,1]$ and $\mu = \mathcal{L}^1$. Give an example in which Vitali's Theorem can be applied but Dominated Convergence Theorem cannot, i.e., a dominating function does not exist.

Hint: look at the functions $f_n^k(x) = \frac{1}{x}\chi_{[\frac{n+k-1}{n2^{n+1}}, \frac{n+k}{n2^{n+1}})}(x)$ for $n \in \mathbb{N}$, $1 \leq k \leq n$.