Exercise 13.1. ♣

(a) The value of the limit

$$\lim_{k\to\infty}\int_0^k \left(1-\frac{x}{k}\right)^k e^{x/3}\,dx$$

is

(A) 0. (B) $\frac{3}{2}$. (C) 1. (D) $+\infty$.

(b) The value of the limit

$$\lim_{n\to\infty} n\int_0^\infty \frac{\sin\left(\frac{x}{n}\right)}{x(1+x^2)}\,dx$$

is

(A) 0. (B) $\frac{\pi}{2}$. (C) $\frac{\pi}{4}$. (D) 1.

(c) Let $f_n \in L^1(0,1)\cap L^2(0,1)$ for $n=1,2,3,\dots$ and consider the following statements:

(i) If $\|f_n\|_{L^1}\to 0$, then $\|f_n\|_{L^2}\to 0$.

(ii) If $\|f_n\|_{L^2}\to 0$, then $\|f_n\|_{L^1}\to 0$.

Which of them are true?

(A) Both (i) and (ii).

(B) (i) but not (ii).

(C) (ii) but not (i).

(D) Neither (i) nor (ii).

(d) Is it true that a sequence of functions in $L^1(0,1)$ converging in measure also converges in the L^1 norm?

Exercise 13.2.

Consider the functions

$$f_n(x)=\sqrt{n}\chi_{[\log(n),\log(n+1)]}(x)$$

defined on $(0,\infty)$. Determine the values of $p\in[1,+\infty]$ such that $f_n\to 0$ in L^p as $n\to\infty$.

Exercise 13.3.
Let $f \in L^p(\mathbb{R}, \lambda)$, where λ is the Lebesgue measure. By means of Fubini's Theorem, show that the following equality holds:

$$\int_{\mathbb{R}} |f(x)|^p dx = p \int_0^\infty y^{p-1} \lambda(\{x \in \mathbb{R} : |f(x)| \geq y\})\, dy.$$

Hint: $|f(x)|^p = \int_0^{|f(x)|} p y^{p-1} dy$.

Exercise 13.4.
Define the function $f : [0,1]^2 \to \mathbb{R}$ as

$$f(x,y) := \begin{cases} y^{-2} & \text{if } 0 < x < y < 1, \\ -x^{-2} & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Is this function summable with respect to the Lebesgue measure?

Exercise 13.5.
Let $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^n)$ and, for all $h \in \mathbb{R}^n$, consider the function $\tau_h \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\tau_h(x) = x + h$. Show that

$$\|f \circ \tau_h - f\|_{L^p} \to 0 \qquad \text{as } h \to 0.$$

Hint: use the density of continuous and compactly supported functions in L^p (Theorem 3.7.15 in the Lecture Notes).

Exercise 13.6. ★
We say that a family $(\varphi_\varepsilon)_{\varepsilon>0}$ of functions in $L^1(\mathbb{R}^n)$ is an *approximate identity* if:

1. $\varphi_\varepsilon \geq 0$ and $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = 1$ for all $\varepsilon > 0$;
2. for all $\delta > 0$ we have that $\int_{\{|x|\geq\delta\}} \varphi_\varepsilon(x)dx \to 0$ as $\varepsilon \to 0$.

(a) Given $\varphi \in L^1(\mathbb{R}^n)$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ for all $\varepsilon > 0$. Show that $(\varphi_\varepsilon)_{\varepsilon>0}$ is an approximate identity.

Let $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^n)$ be an approximate identity. Show that the following statements hold.

(b) If $f \in L^\infty(\mathbb{R}^n)$ is continuous at $x_0 \in \mathbb{R}^n$, then $f*\varphi_\varepsilon$ is continuous and $(f*\varphi_\varepsilon)(x_0) \to f(x_0)$ as $\varepsilon \to 0^+$.

(c) If $f \in L^\infty(\mathbb{R}^n)$ is uniformly continuous, then $f * \varphi_\varepsilon \xrightarrow{L^\infty} f$ as $\varepsilon \to 0^+$.

(d) If $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^n)$, then $f * \varphi_\varepsilon \xrightarrow{L^p} f$ as $\varepsilon \to 0^+$.
Hint: use Hölder's inequality and keep in mind Exercise 13.5 and part (b).

Exercise 13.7.
Compute the following limits:
(a)
$$\lim_{n\to\infty} \int_0^1 \frac{1+nx}{(1+x)^n}\,dx.$$
(b)
$$\lim_{n\to\infty} \int_0^1 \frac{x\log x}{1+n^2x^2}\,dx.$$

Exercise 13.8.
Let $I = [0,1]$ and consider the function
$$f : I^3 \to [0,\infty], \quad f(x,y,z) := \begin{cases} \frac{1}{\sqrt{|y-z|}}, & \text{if } y \neq z, \\ \infty, & \text{if } y = z. \end{cases}$$
Show that $f \in L^1(I^3, \mathcal{L}^3)$.

Exercise 13.9. ★

The goal of this exercise is to construct an $\mathcal{L}^1$-measurable set $A \subset [0,1]$ with the property that both

$$\mathcal{L}^1(U \cap A) > 0 \quad \text{and} \quad \mathcal{L}^1(U \setminus A) > 0 \tag{$*$}$$

for every nonempty open subset $U \subset [0,1]$.

(a) Show that it is enough to check $(*)$ for dyadic intervals U, that is, for sets U of the form $U = (m2^{-j}, (m+1)2^{-j})$ with integers $j \geq 1$ and $0 \leq m < 2^j$.

The idea of the proof will be to modify iteratively our set by small amounts, so that its measure in all dyadic intervals of smaller and smaller sizes is controlled from above and below. This is the main construction that we will need in the iteration:

(b) Show that given any measurable set $E \subset [0,1]$, any integer $k \geq 1$ and any real number $0 < \beta \leq 2^{-(k+1)}$, one can find a measurable set $B \subset [0,1]$ such that

$$\mathcal{L}^1((m2^{-k}, (m+1)2^{-k}) \cap B) \geq \beta \;\text{ and }\; \mathcal{L}^1((m2^{-k}, (m+1)2^{-k}) \setminus B) \geq \beta \tag{1}$$

for $m = 0, 1, \ldots, 2^k - 1$, and

$$\mathcal{L}^1(E \triangle B) \leq 2^k \beta. \tag{2}$$

Let us now fix a sequence of positive real numbers $\beta_1, \beta_2, \ldots$ satisfying the following condition:

$$\forall k \geq 1 \qquad 2^{-(k+1)} \geq \beta_k > 2^{k+1}\beta_{k+1} + 2^{k+2}\beta_{k+2} + 2^{k+3}\beta_{k+3} + \cdots . \tag{C}$$

We construct inductively using part (b) a sequence of measurable sets $A_0, A_1, A_2, \ldots \subset [0,1]$ with $A_0 = \varnothing$ satisfying the following two properties:

$$\mathcal{L}^1((m2^{-k}, (m+1)2^{-k}) \cap A_k) \geq \beta_k \quad \text{and} \quad \mathcal{L}^1((m2^{-k}, (m+1)2^{-k}) \setminus A_k) \geq \beta_k$$

for $m = 0, 1, \ldots, 2^k - 1$, and

$$\mathcal{L}^1(A_{k-1} \triangle A_k) \leq 2^k \beta_k.$$

(c) Show that there exists a measurable set $A \subset [0,1]$ such that $\mathcal{L}^1(A_k \triangle A) \to 0$ as $k \to \infty$.

Hint: Use the completeness of L^1.

(d) Show that $(*)$ holds for this set A and any dyadic interval U.

(e) To complete the proof, show that if we choose $\beta_k = 2^{-3^k}$, then condition (C) holds.