Exercise 1. (10 points)
Let $\mathcal{A}$ be an algebra of sets on a set X and $\lambda : \mathcal{A} \to [0, +\infty]$ a pre-measure on $\mathcal{A}$. We define the mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ as

$$\mu(E) := \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) \,\middle|\, A_1, A_2, \ldots \in \mathcal{A}, E \subseteq \bigcup_{k=1}^{\infty} A_k \right\}$$

for any set $E \subseteq X$. One can then show that μ is a measure.

(a) (2 points) State Carathéodory's measurability criterion with respect to μ.

Solution: Carathéodory's criterion states that a set $A \subseteq X$ is measurable if and only if for any set $E \subseteq X$,

$$\mu(E) = \mu(E \setminus A) + \mu(E \cap A)$$

holds.

(b) (4 points) Show that for every $A \in \mathcal{A}$, it holds that $\lambda(A) = \mu(A)$.

Solution: Since $\{A, \varnothing, \varnothing, \ldots\}$ is a covering of A by sets in $\mathcal{A}$ and $\lambda(\varnothing) = 0$, it clearly holds that $\mu(A) \leq \lambda(A)$.

For the reverse inequality, let $A_1, A_2, \ldots \in \mathcal{A}$ be such that $A \subseteq \bigcup_{k=1}^{\infty} A_i$ and define the sets $B_k := A_k \cap A \setminus (A_1 \cup \cdots \cup A_{k-1})$. Since $A \in \mathcal{A}$, by the properties of an algebra we see that $B_k \in \mathcal{A}$. Moreover they are pairwise disjoint by construction and satisfy

$$\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A \cap A_k \setminus (A_1 \cup \cdots \cup A_{k-1}) = \bigcup_{k=1}^{\infty} A \cap A_k = A \cap \bigcup_{k=1}^{\infty} A_k = A.$$

Therefore $\lambda(A) = \sum_{k=1}^{\infty} \lambda(B_k) \leq \sum_{k=1}^{\infty} \lambda(A_k)$. Taking the infimum over all such collections we deduce that $\lambda(A) \leq \mu(A)$.

(c) (4 points) Prove that every set $A \in \mathcal{A}$ is μ-measurable.

Solution: Let $E \in \mathcal{P}(X)$ be arbitrary. The inequality $\mu(E) \leq \mu(E \setminus A) + \mu(E \cap A)$ is easy and can be taken for granted.

For the opposite one, let $A_1, A_2, \ldots \in \mathcal{A}$ cover E and observe that the collections $\{A_i \setminus A\}$ and $\{A_i \cap A\}$ are covers of $E \setminus A$ and $E \cap A$ (respectively) also by sets in $\mathcal{A}$. Then, using the additivity of λ,

$$\mu(E \setminus A) + \mu(E \cap A) \leq \sum_{k=1}^{\infty} \lambda(A_k \setminus A) + \lambda(A_k \cap A) = \sum_{k=1}^{\infty} \lambda(A_k).$$

We now finish by taking the infimum over all such collections $\{A_i\}$.

Note: You do **not** need to show that μ is a measure but you **are allowed** to use that it satisfies the properties of a measure as long as you state them clearly.

Exercise 2. (14 points)

(a) (2 points) Let (Ω, μ) be a measure space. For $1 \leq p < +\infty$, define the space $L^p(\Omega, \mu)$.

Solution: For $1 \leq p < +\infty$, we define $L^p(\Omega, \mu)$ to be the space of measurable functions $f : \Omega \to \overline{\mathbb{R}}$ such that

$$\|f\|^p_{L^p(\Omega,\mu)} := \int_\Omega |f|^p \, d\mu < +\infty,$$

up to almost everywhere equivalence.

Fix $1 \leq p < +\infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\Omega, \mu)$.

(b) (4 points) Show that there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \|f_{n_k} - f_{n_{k+1}}\|_{L^p(\Omega,\mu)} < +\infty.$$

Solution: We take inducively n_k to be bigger than n_{k-1} and to satisfy that $\forall n \geq n_k$, $\|f_n - f_{n_k}\|_{L^p} \leq 2^{-k}$. This is possible thanks to the fact that $\{f_n\}$ is a Cauchy sequence in L^p. The desired sum clearly converges with this choice.

(c) (4 points) Show that the function $g(x) := \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ is in $L^p(\Omega, \mu)$ and is finite μ-almost everywhere.

Solution: Let $g_K(x) := \sum_{k=1}^{K} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ and observe that $g_K \nearrow g$ monotonically almost everywhere. On the other hand, by the Minkowski inequality,

$$\|g_K\|_{L^p} \leq \sum_{k=1}^{K} \|f_{n_{k+1}}(x) - f_{n_k}(x)\|_{L^p} \leq C,$$

where C is the value of the finite sum in part (a). Observe that also $g_K^p \nearrow g^p$ monotonically almost everywhere, thus by Beppo Levi's theorem,

$$\int_\Omega g^p = \lim_{k\to\infty} \int_\Omega g_k^p = \lim_{k\to\infty} \|g_k\|^p_{L^p} \leq C^p < +\infty.$$

Hence g is in L^p and in particular finite almost everywhere.

(d) (4 points) Prove that there exists a function $f \in L^p(\Omega, \mu)$ such that $\|f_{n_k} - f\|_{L^p(\Omega,\mu)} \to 0$ as $k \to \infty$ and deduce that $L^p(\Omega, \mu)$ is complete.

Solution: We have seen that for μ-almost every $x \in \Omega$ the sum $\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ converges. Since an absolutely converging series of real numbers is converging, this means that the sequence $f_{n_K}(x) = f_{n_1}(x) + \sum_{k=1}^{K-1} \left(f_{n_{k+1}}(x) - f_{n_k}(x) \right)$ converges for almost every x to a value that we call $f(x)$. As the limit of measurable functions f is measurable. Moreover,

$$|f| \leq |f_{n_1}| + g \in L^p(\Omega, \mu)$$

and by applying dominated convergence to the functions $|f - f_{n_K}|^p \leq (|f| + |f_{n_K}|)^p \leq (|f| + g + |f_{n_1}|)^p \in L^1$ we deduce that

$$\lim_{K \to \infty} \|f - f_{n_K}\|_{L^p}^p = 0.$$

Finally, since the whole sequence $\{f_n\}$ is Cauchy and a subsequence of it converges, all the sequence must converge as well. Thus completeness is proven.

Exercise 3. (6 points)
Compute the following limits:

(a) (3 points)

$$\lim_{m\to\infty}\int_0^\infty \frac{1}{1+x^m}\,dx$$

Solution: Observe that for $0 \le x < 1$, $x^m \to 0$ as $m \to \infty$, whereas for $x > 1$, $x^m \xrightarrow{m\to\infty} +\infty$. Hence, if we define $f_m(x)$ to be the integrand, $f_m \to \chi_{[0,1]}$ almost everywhere. In order to apply Lebesgue's dominated convergence theorem, we need to find a dominating function. For $x > 1$ and $m \ge 2$, observe that $x^2 \le x^m$, which implies that $f_m(x) \le \frac{1}{1+x^2}$. On the other hand, for $x \le 1$, $f_m(x) \le 1$. Thus the function $g(x) = \chi_{[0,1]}(x) + \frac{1}{1+x^2}$ dominates $\{f_m\}$ and is summable, hence we can pass to the limit

$$\lim_{m\to\infty}\int_0^\infty f_m(x)\,dx = \int_0^\infty \chi_{[0,1]}(x)\,dx = 1.$$

(b) (3 points)

$$\lim_{m\to\infty}\int_0^1 \sum_{k=0}^m \frac{x^k}{k!}\,dx$$

Solution: Since the summands are positive, the sum inside the integral is a monotone function of m for each x. Therefore we can apply Beppo Levi's monotone convergence theorem and deduce

$$\lim_{m\to\infty}\int_0^1 \sum_{k=0}^m \frac{x^k}{k!}\,dx = \int_0^1 \sum_{k=0}^\infty \frac{x^k}{k!}\,dx = \int_0^1 e^x\,dx = e^1 - e^0 = e - 1.$$