Exercise 2.1.
Given a measure μ on a set X, we define the set of atoms of μ as

$$A_\mu := \{x \in X : \{x\} \text{ is measurable and } \mu(\{x\}) > 0\}.$$

(a) Assuming that $\mu(X) < +\infty$, show that A_μ is at most countable.

Solution: For each $n \in \mathbb{Z}^+$ let $A(n)$ be the set of $x \in X$ such that $\{x\}$ is measurable and $\mu(\{x\}) \geq \frac{1}{n}$. We will show that $\#(A(n)) \leq n\mu(X) < +\infty$. Indeed, if it were not the case we could find a finite set $F \subset A(n)$ with more than $n\mu(X)$ elements, to which we can apply the additivity of μ:

$$\mu(X) < \frac{1}{n}\#(F) \leq \sum_{x \in F} \mu(\{x\}) = \mu\left(\bigcup_{x \in F} \{x\}\right) = \mu(F) \leq \mu(X) < +\infty,$$

which is a contradiction. Therefore $A(n)$ is finite for each n, and $A_\mu = \bigcup_{n \in \mathbb{Z}^+} A(n)$ is a countable union of finite sets, thus is countable. □

(b) Is the same true if μ is only assumed to be σ-finite? And in general? Show it or give a counterexample.

Solution: The statement still holds true X is σ-finite: one can write $X = \bigcup_{k \in \mathbb{Z}^+} X_k$ with $X_k \subset X$ μ-measurable and $\mu(X_k) < +\infty$ and deduce as above that for each k, the set $A_\mu \cap X_k$ is finite. Then

$$A_\mu = \bigcup_{k \in \mathbb{Z}^+} (A_\mu \cap X_k)$$

is a countable union of countable sets and therefore countable.

However without the σ-finiteness assumption the statement is false: consider any uncountable set X (for example $X = \mathbb{R}$) together with the counting measure $\#$. Then every singleton is measurable and has positive measure. □

(c) ★ Construct an example of measure μ on an uncountable set X such that $\mu(\{x\}) > 0$ for every $x \in X$ but $\mu(X) < \infty$. This shows that the condition of the measurability of $\{x\}$ in the definition of A_μ cannot be removed.

Solution: Let X be any uncountable set (for example $X = \mathbb{R}$) and consider the measure μ defined by

$$\mu(E) = \begin{cases} 0, & E = \varnothing \\ 1, & E \neq \varnothing \end{cases}$$

(see Exercise 1.2.13, part 3, in the lecture notes.) Then $\mu(X) = 1 < \infty$ but $\mu(\{x\}) = 1 > 0$ for every $x \in X$. However only the sets $\varnothing$ and X are measurable. □

Exercise 2.2. (Upper and lower semicontinuity of measures.)
Let $\mathcal{E}$ be a σ-algebra on a set X and $\mu : \mathcal{E} \to [0, \infty]$ a σ-additive function on $\mathcal{E}$. For a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{E}$,

(a) show that

$$\mu\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mu(A_n).$$

Solution: Recall that $\liminf_{n\to\infty} A_n$ is the increasing union $\bigcup_{n\geq 1}\bigcap_{m\geq n} A_m$. Therefore it holds that

$$\mu\left(\liminf_{n\to\infty} A_n\right) = \mu\left(\bigcup_{n\geq 1}\bigcap_{m\geq n} A_m\right) = \lim_{n\to\infty}\mu\left(\bigcap_{m\geq n} A_m\right).$$

On the other hand, since $\bigcap_{m\geq n} A_m \subseteq A_n$, it holds that $\mu(\bigcap_{m\geq n} A_m) \leq \mu(A_n)$, thus passing to the lim inf we get

$$\mu\left(\liminf_{n\to\infty} A_n\right) = \lim_{n\to\infty}\mu\left(\bigcap_{m\geq n} A_m\right) \leq \liminf_{n\to\infty}\mu(A_n). \qquad \square$$

(b) show that also

$$\limsup_{n\to\infty}\mu(A_n) \leq \mu\left(\limsup_{n\to\infty} A_n\right)$$

holds provided that $\mu(X) < \infty$.

Solution: Similarly, $\limsup_{n\to\infty} A_n$ is the decreasing intersection $\bigcap_{n\geq 1}\bigcup_{m\geq n} A_m$. Assuming that $\mu(X) < \infty$, it holds that

$$\mu\left(\limsup_{n\to\infty} A_n\right) = \mu\left(\bigcap_{n\geq 1}\bigcup_{m\geq n} A_m\right) = \lim_{n\to\infty}\mu\left(\bigcup_{m\geq n} A_m\right).$$

Analogously, since $\bigcup_{m\geq n} A_m \supseteq A_n$, we see that $\mu(\bigcup_{m\geq n} A_m) \geq \mu(A_n)$, thus we can take the lim sup on both sides and conclude

$$\mu\left(\limsup_{n\to\infty} A_n\right) = \lim_{n\to\infty}\mu\left(\bigcup_{m\geq n} A_m\right) \geq \limsup_{n\to\infty}\mu(A_n). \qquad \square$$

Exercise 2.3. ♣

Let $\mathcal{E}$ be a σ-algebra on a set X and $\mu : \mathcal{E} \to [0,\infty]$ a σ-additive function on $\mathcal{E}$. Which of the following are true for an arbitrary sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$?

(a) Whenever $B \subseteq \bigcup_{n=1}^{\infty} A_n$,

$$\mu(B) \leq \sum_{n=1}^{\infty}\mu(A_n). \quad ✓$$

(b)

$$\limsup_{n\to\infty} A_n^c = \left(\liminf_{n\to\infty} A_n\right)^c. \quad ✓$$

(c)

$$\mu\left(\liminf_{n\to\infty} A_n\right) = \liminf_{n\to\infty}\mu(A_n). \quad ✗$$

(d)

$$\limsup_{n\to\infty} \mu(A_n) \leq \mu\left(\limsup_{n\to\infty} A_n\right). \quad ✗$$

Exercise 2.4. ★
Let $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ be a measure with the following property: there is a real number $s > n$ such that for every $x \in \mathbb{R}^n$ and $r > 0$,

$$\mu(B(x, r)) \leq r^s.$$

Show that $\mu \equiv 0$. Here $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ denotes the open ball with center x and radius r in $\mathbb{R}^n$.

Solution: The key idea is to use cubes instead of squares. Given a cube $Q = [a_1, a_1 + \ell] \times \cdots \times [a_n, a_n + \ell] \subset \mathbb{R}^n$, it is easy to see that it is contained in the ball $B(x, r)$ where $x = (a_1 + \ell/2, \ldots, a_n + \ell/2)$ and r is any number bigger than $\sqrt{n}\ell/2$. Therefore, by the monotonicity property of μ, our assumption implies that

$$\mu(Q) \leq \mu(B(x, r)) \leq \left(\frac{\sqrt{n}\ell}{2}\right)^s = C\ell^s,$$

where C is just a constant that depends on n and s.

Now given any cube $Q = [a_1, a_1 + \ell] \times \cdots \times [a_n, a_n + \ell]$ and any number k, decompose Q into k^n cubes of sidelength ℓ/k in the obvious way. That is, the new cubes will be indexed by n integers $i_1, \ldots, i_n \in \{0, \ldots, k-1\}$ and

$$Q_{i_1,\ldots,i_n} = [a_1 + i_1\ell/k, a_1 + (i_1 + 1)\ell/k] \times \cdots \times [a_n + i_n\ell/k, a_n + (i_n + 1)\ell/k].$$

Hence by the subadditivity of the measure,

$$\mu(Q) \leq \sum_{(i_1,\ldots,i_n)\in\{0,\ldots,k-1\}^n} \mu(Q_{i_1,\ldots,i_n}) \leq \sum_{(i_1,\ldots,i_n)\in\{0,\ldots,k-1\}^n} C\ell^s = k^n \left(\frac{\ell}{k}\right)^s = k^{n-s}\ell^s.$$

Letting $k \to \infty$ the right hand side tends to zero because $s > n$. Hence any cube has measure zero, and since $\mathbb{R}^n$ is a union of cubes, $\mu \equiv 0$. □