Exercise 3.1.
Denote by λ the Lebesgue measure on $\mathbb{R}$. Let $E \subset [0,1]$ be a Lebesgue measurable set of strictly positive measure, i.e. $\lambda(E) > 0$. Show that for any $0 \leq \delta \leq \lambda(E)$, there exists a measurable subset of E having measure exactly δ.

Hint: Consider the function $t \in [0,1] \mapsto \lambda([0,t] \cap E)$.

Solution: Consider the following function $f : [0,1] \to \mathbb{R}$:

$$f(t) = \lambda([0,t] \cap E), \quad t \in [0,1].$$

Notice that $f(0) = 0$ as well as $f(1) = \lambda(E)$. We want to show that f is continuous. Therefore, take $0 \leq s < t \leq 1$. Due to the additivity on the disjoint subsets $[0,s] \cap E$ and $(s,t] \cap E$, it holds:

$$f(t) = \lambda([0,t] \cap E) = \lambda([0,s] \cap E) + \lambda((s,t] \cap E) \leq f(s) + t - s$$

where the monotonicity of λ was used in the last inequality. Consequently, we conclude

$$|f(t) - f(s)| \leq |t - s|,$$

which implies continuity.

The Intermediate Value Theorem states that for any δ between 0 and $\lambda(E)$, there exists a point x such that $f(x) = \delta$. As a result, the set $[0,x] \cap E$ (measurable as an intersection of measurable subsets) satisfies the desired property:

$$\lambda([0,x] \cap E) = \delta. \qquad \square$$

Exercise 3.2.
Let $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ be the function

$$\mu(A) := \sqrt{\mathcal{L}^1(A)}$$

for $A \subseteq \mathbb{R}$, where $\mathcal{L}^1$ denotes the Lebesgue measure.

(a) Show that μ is a measure.

Solution: Clearly $\mu(\varnothing) = 0$. Let $A \subseteq \bigcup_{k=1}^{\infty} A_k$. Then, by the subadditivity of $\mathcal{L}^1$,

$$\mathcal{L}^1(A) \leq \sum_{k=1}^{\infty} \mathcal{L}^1(A_k).$$

Since for every $m \in \mathbb{N}$ we have

$$\sum_{k=1}^{m} \mathcal{L}^1(A_k) \leq \sum_{k=1}^{m} \mathcal{L}^1(A_k) + \sum_{1 \leq j < k \leq m} 2\mathcal{L}^1(A_j)^{1/2}\mathcal{L}^1(A_k)^{1/2} = \left(\sum_{k=1}^{m} \mathcal{L}^1(A_k)^{1/2}\right)^2 \leq \left(\sum_{k=1}^{\infty} \mathcal{L}^1(A_k)^{1/2}\right)^2,$$

passing to the limit we obtain that

$$\mathcal{L}^1(A) \leq \sum_{k=1}^{\infty} \mathcal{L}^1(A_k) \leq \left(\sum_{k=1}^{\infty} \mathcal{L}^1(A_k)^{1/2} \right)^2 ,$$

which is what we wanted to show.

(b) ★ What is the σ-algebra of μ-measurable sets?

Solution: Let $A \subset \mathbb{R}$ be such that $\mathcal{L}^1(A) > 0$ and $\mathcal{L}^1(\mathbb{R} \setminus A) > 0$. We claim that A is not μ-measurable. Indeed, by the previous exercise we can choose sets $B \subseteq A$ and $C \subseteq \mathbb{R} \setminus A$ such that $0 < \mathcal{L}^1(B), \mathcal{L}^1(C) < \infty$. Then testing measurability against $E := B \cup C$ we find

$$\mu(E) = \sqrt{\mathcal{L}^1(E)} \leq \sqrt{\mathcal{L}^1(B) + \mathcal{L}^1(C)} < \sqrt{\mathcal{L}^1(B)} + \sqrt{\mathcal{L}^1(C)} = \mu(E \cap A) + \mu(E \setminus A) < \infty$$

where we have used that $E \cap A = B$ and $E \setminus A = C$, and that the inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ is strict whenever x, y are positive finite real numbers.

On the other hand, every set A with $\mathcal{L}^1(A) = 0$ or $\mathcal{L}^1(\mathbb{R} \setminus A) = 0$ has zero μ-measure (or its complement does) so it is automatically μ-measurable. Thus the σ-algebra of μ-measurable sets consists precisely of the Lebesgue-null sets and their complements.

Exercise 3.3.

Recall that the system of elementary sets is defined as

$$\mathcal{A} := \{A \subset \mathbb{R}^n \mid A \text{ is the union of finitely many disjoint intervals}\}.$$

(a) Prove that $\mathcal{A}$ is an algebra. To simplify the notation you may assume that $n = 1$.

Solution: To prove that the collection of elementary sets $\mathcal{A}$ is an algebra, we need to show that $\mathbb{R}^n \in \mathcal{A}$ as well as the closedness of $\mathcal{A}$ with respect to taking complements and finite unions.

It is easy to see that $\mathbb{R}^n$ is an interval (see definition in Lecture Notes, $a, b = \pm\infty$ is allowed). Therefore, it belongs to $\mathcal{A}$. Let now $A = \bigcup_{k=1}^m A_k$ where A_k are disjoint intervals. The complements of the A_k can be expressed as:

$$A_k^c = \bigcup_{i=1}^{p(n)} B_{k,i}$$

where $p(n)$ depends on the dimension of $\mathbb{R}^n$ and determines how many pieces are needed to express the complement as a union of intervals. As a result, A_k^c is again in $\mathcal{A}$. Now, using de Morgan, we see:

$$A^c = \left(\bigcup_{k=1}^m A_k \right)^c = \bigcap_{k=1}^m A_k^c ,$$

However, it is obvious (since the intersection of two intervals is another interval), that the intersection of two sets $A, B \in \mathcal{A}$ lies again in $\mathcal{A}$. Therefore, we have shown $A^c \in \mathcal{A}$.

Finally, let $A_k = \bigcup_{l=1}^{n_k} A_{kl} \in \mathcal{A}$ where A_{kl} are pairwise disjoint intervals for $k = 1, \ldots, m$ and $l = 1, \ldots, n_k$. In this case, $\bigcup_{k=1}^m A_k = \bigcup_{k=1}^m \bigcup_{l=1}^{n_k} A_{kl}$ is a finite union of intervals. In addition, they can be chosen to be disjoint. To see this, let us consider the case $m = 2$, the general case follows by repeated application of the case $m = 2$. For all $l \in \{1, \dots, n_2\}$, we define:

$$\tilde{A}_{2l} := A_{2l} \setminus \bigcup_{j=1}^{n_1} A_{1j} = A_{2l} \cap \bigcap_{j=1}^{n_1} A_{1j}^c.$$

As we argued before, A_{1j}^c is again an elementary subset and the finite intersection of elementary subsets is again elementary (consider their decomposition into disjoint cubes to see this). Therefore, $\tilde{A}_{2l}$ is elementary. Moreover, observe that all $\tilde{A}_{2l}$ are pairwise disjoint with each other and each of the A_{1j}. Therefore, using their decomposition into disjoint cubes, we can deduce that:

$$A_1 \cup A_2 \in \mathcal{A}.$$

Consequently, we see $\bigcup_{k=1}^m A_k \in \mathcal{A}$. This yields that $\mathcal{A}$ is an algebra. □

A more direct proof can be given as follows: again, we are given a finite collection of elementary sets $A_1, \dots, A_m$. For each $1 \le k \le n$, let

$$-\infty =: a_0^k < a_1^k < a_2^k < \ldots < a_{q_k-1}^k < a_{q_k}^k := +\infty$$

be the finite collection of numbers which appear as one of the endpoints of the k-th factor of one of the intervals that constitute one of the A_j.

Namely, for each $k \in \{1, \dots, n\}$, let S^k be the union of the sets of endpoints of all the intervals I_k that are the k-th factor of one of the $I \subset A_j$, together with $\pm\infty$. Since S^k is finite, we can write it as $\{a_0^k, \dots, a_{q_k}^k\}$, where the elements are ordered increasingly and $a_0^k = -\infty, a_{q_k}^k = +\infty$.

Consider then the finite collection of intervals

$$\mathcal{J} = \left\{ J_1 \times \cdots \times J_n \,\middle|\, \text{for each } k, J_k = \{a_i^k\} \text{ with } 0 < i < q_k \text{ or } J_k = (a_i^k, a_{i+1}^k) \text{ with } 0 \le i < q_k \right\}.$$

It is clear that the intervals in $\mathcal{J}$ partition $\mathbb{R}^n$, and that each A_j is a union of intervals in $\mathcal{J}$. Therefore the union $A_1 \cup \dots \cup A_m$ can be written as the (disjoint) union of those $J \in \mathcal{J}$ which are contained in some A_j, and in particular is a disjoint union of intervals.

Note also that this also shows at once that $\mathcal{A}$ is closed under complements: defining $\mathcal{J}$ as above only for the elementary set A, we see that A is the union of some intervals from $\mathcal{J}$, therefore A^c is the union of the remaining ones.

(b) ★ Show that the volume function vol introduced in the lecture[1] for elementary sets is a pre-measure.

Remark: For $I = I_1 \times \ldots \times I_n$ an interval in $\mathbb{R}^n$, its volume is defined by

$$\operatorname{vol}(I) = \prod_{k=1}^n \operatorname{vol}(I_k),$$

where for an interval $I_k \subseteq \mathbb{R}$, $\operatorname{vol}(I_k)$ is the length of I_k.

[1] Definition 1.3.1 in the Lecture Notes.

Solution: Let $\{A_k\}_{k\in\mathbb{N}}$ be a countable, pairwise disjoint family of elementary sets and assume that $A = \bigcup_{k=1}^{\infty} A_k$ is another elementary set. We want to show:

$$\mathrm{vol}(A) = \sum_{k=1}^{\infty} \mathrm{vol}(A_k).$$

First of all, by considering instead of A_k its building blocks, we can assume that each A_k is an interval. The $\geq$ inequality is easy to see because

$$\mathrm{vol}(A) \geq \sum_{k=1}^{m} \mathrm{vol}(A_k)$$

holds for each m due to the monotonicity of the volume.

For the opposite inequality, let $\varepsilon > 0$ and take a compact elementary set $B \subset A$ such that $\mathrm{vol}(A) \leq \mathrm{vol}(B) + \varepsilon$ if $\mathrm{vol}(A) < \infty$ or such that $\mathrm{vol}(B) \geq \varepsilon^{-1}$ if $\mathrm{vol}(A) = \infty$. Then take open intervals U_k containing $A_k \cap B$ with $\mathrm{vol}(U_k) \leq \mathrm{vol}(A_k \cap B) + 2^{-k}\varepsilon$. All of these sets are easy to construct by slightly changing the endpoints of the intervals.

Since B is a compact set covered by the open sets U_k, we can extract a finite cover $U_{k_1}, \dots, U_{k_m}$ of B and therefore

$$\mathrm{vol}(B) \leq \sum_{i=1}^{m} \mathrm{vol}(U_{k_i}) \leq \sum_{k=1}^{\infty} \mathrm{vol}(U_k) \leq \sum_{k=1}^{\infty} \left(\mathrm{vol}(A_k \cap B) + 2^{-k}\varepsilon\right) \leq \sum_{k=1}^{\infty} \mathrm{vol}(A_k) + \varepsilon.$$

Letting now $\varepsilon \to 0$, the left hand side converges to $\mathrm{vol}(A)$ and the right hand side converges to the sum $\sum_{k=1}^{\infty} \mathrm{vol}(A_k)$, thus proving the $\leq$.

Exercise 3.4.

(a) Let X be any set with more than one element and consider the measure $\mu : \mathcal{P}(X) \to [0, +\infty]$ defined by:

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{else} \end{cases}.$$

Give an example of a non-μ-measurable subset.

Solution: We prove that A is μ-measurable if and only if $A \in \{\emptyset, X\}$. Assume that $A \neq \emptyset, X$. In this case, there exist $x, y \in X$ such that $x \in A$ und $y \in A^c$. Consequently, we deduce:

$$\mu(X) = 1 \neq 2 = \mu(X \setminus A) + \mu(X \cap A).$$

As a result, any subset $A \neq \emptyset, X$ is not μ-measurable. Conversely, it is immediate to check that $\emptyset$ and X are μ-measurable. □

(b) Define $\mathcal{A}$ to be the algebra in $\mathbb{R}$ generated by the half-closed intervals of the form $[a, b[$ for every $-\infty \leq a < b \leq \infty$. Note that any element in $\mathcal{A}$ can be expressed as the disjoint union of finitely many intervals of the type described before. Moreover, we define:

$$\lambda : \mathcal{A} \to [0, +\infty], \quad \lambda(A) = \begin{cases} +\infty & \text{if } A \neq \emptyset \\ 0 & \text{else} \end{cases}$$

Check that λ is a pre-measure and find two distinct Carathéodory-Hahn extensions of λ, i.e. two measures on $\mathcal{P}(\mathbb{R})$ which coincide with λ on $\mathcal{A}$. Why does this not yield a contradiction to the uniqueness statement Theorem 1.2.21 of the Lecture Notes?

Solution: Checking that λ is a pre-measure is trivial. Now consider the counting measure $\mu_1 : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, i.e.

$$\mu_1(A) = \begin{cases} \#A & \text{if } A \text{ is finite} \\ +\infty & \text{otherwise,} \end{cases}$$

and the measure $\mu_2 : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ given by

$$\mu_2(A) = \begin{cases} +\infty & \text{if } A \text{ has uncountably many elements} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that μ_1 and μ_2 are distinct extensions of λ that do not agree on the σ-algebra generated by $\mathcal{A}$, which is the Borel σ-algebra of $\mathbb{R}$ (see Example 1.1.9 of the Lecture Notes). This does not contradict Theorem 1.2.21 in the Lecture Notes due to the simple observation that λ is not σ-finite. □

Exercise 3.5. ♣

Let μ be a measure on $\mathbb{R}^n$ and $A, B_1, B_2, \ldots \subset \mathbb{R}^n$ be such that $A \subseteq \limsup_{k\to\infty} B_k$ and $\sum_{k=1}^{\infty} \mu(B_k) < \infty$. Which of the following statements are true?

(a) $\mu(A) > 0$. ✗

(b) $\mu(A) = 0$. ✓

(c) Every point of A belongs to infinitely many of the B_k. ✓

(d) Every point of A belongs to all except possibly finitely many of the B_k. ✗