Exercise 5.1.
Fix some $0 < \beta < 1/3$ and define $I_1 = [0,1]$. For every $n \geq 1$, let $I_{n+1} \subset I_n$ be the collection of intervals obtained removing from every interval in I_n its centered open subinterval of length β^n. Then define by $C_\beta = \bigcap_{n=1}^\infty I_n$, the *fat Cantor set* corresponding to β.

Show that:

(a) C_β is Lebesgue measurable with measure $\mathcal{L}^1(C_\beta) = 1 - \frac{\beta}{1-2\beta}$.

Solution: The set I_n is Lebesgue measurable, since it consists of 2^{n-1} intervals, and has measure $\mathcal{L}^1(I_n) = \mathcal{L}^1(I_{n-1}) - 2^{n-2}\beta^{n-1}$ for all $n \geq 2$, with $\mathcal{L}^1(I_1) = 1$. Hence

$$\mathcal{L}^1(I_n) = 1 - \sum_{k=1}^{n-1} 2^{k-1}\beta^k = 1 - \frac{1}{2}\sum_{k=1}^{n-1}(2\beta)^k = 1 - \frac{1}{2}\left(\frac{1-(2\beta)^n}{1-2\beta} - 1\right) = 1 - \frac{\beta - 2^{n-1}\beta^n}{1-2\beta}.$$

As a result $C_\beta = \bigcap_{n=1}^\infty I_n$ is Lebesgue measurable with measure

$$\mathcal{L}^1(C_\beta) = \lim_{n\to\infty} \mathcal{L}^1(I_n) = 1 - \frac{\beta}{1-2\beta}. \qquad \square$$

(b) C_β is not Jordan measurable. Indeed it holds $\underline{\mu}(C_\beta) = 0$ and $\overline{\mu}(C_\beta) = 1 - \frac{\beta}{1-2\beta} > 0$.

Solution: First note that C_β has empty interior, which follows from the fact that I_n consists of 2^{n-1} intervals of length $(1 - \frac{\beta}{1-2\beta})2^{-(n-1)} + \frac{\beta^n}{1-2\beta}$, which converges to 0 as $n \to \infty$. Therefore $\underline{\mu}(C_\beta) = 0$. On the other hand $\overline{\mu}(C_\beta) \geq \mathcal{L}^1(C_\beta) = 1 - \frac{\beta}{1-2\beta}$ and this is actually an equality since I_n is an elementary set for all $n \geq 1$ and therefore $\overline{\mu}(C_\beta) \leq \inf_{n\geq 1} \mathcal{L}^1(I_n) = 1 - \frac{\beta}{1-2\beta}$. Hence $\overline{\mu}(C_\beta) = 1 - \frac{\beta}{1-2\beta}$, which is greater than 0 for $0 < \beta < 1/3$. Hence C_β is not Jordan measurable. $\square$

Exercise 5.2.
The goal of this exercise is to show that the Cantor triadic set C is uncountable. For that, recall quickly the construction of C: Every $x \in [0,1]$ can be expanded in base 3, i.e., can be written as $x = \sum_{i=1}^\infty d_i(x)3^{-i}$ for $d_i(x) \in \{0,1,2\}$. The set C is then defined as the set of those $x \in [0,1]$ that do not have any digit 1 in their 3-expansion, i.e.:,

$$C := \{x \in [0,1] \mid d_i(x) \in \{0,2\}, \forall i \in \mathbb{N}\}.$$

Now, the Cantor-Lebesgue function F is defined by

$$F : C \to [0,1], \quad F\left(\sum_{i=1}^\infty \frac{a_i}{3^i}\right) := \sum_{i=1}^\infty \frac{a_i}{2^{i+1}}.$$

(a) Show that $F(0) = 0$ and $F(1) = 1$.

Solution: We see $0 = \sum_{i=1}^\infty 0 \cdot 3^{-i}$ and as a result, $F(0) = \sum_{i=1}^\infty 0 \cdot 2^{-(i+1)} = 0$. For 1 we have the expansion $1 = 0.2222...$, so $1 = \sum_{i=1}^\infty 2 \cdot 3^{-i}$ and therefore

$$F(1) = \sum_{i=1}^\infty 2 \cdot \frac{1}{2^{i+1}} = \frac{1}{2} \cdot \sum_{i=0}^\infty \frac{1}{2^i} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1. \qquad \square$$

(b) Show that F is well-defined and continuous on C.

Solution: In general, expansions in the base 3 of an element $x \in [0,1]$ are not unique, see for example $0.1 = 0.022222.....$ However, if we restrict ourselves to expansions only using the coefficients 0 and 2, the expansion becomes unique, which shows that F is well-defined on C. (It could be easily shown that F would even be well-defined on $[0,1]$ by investigating periodic expansions more closely).

We now proceed to show that F is continuous on C. Let $\varepsilon > 0$. Take any $x \in C$ and $\{x_n\}_{n=0}^{\infty}$ any sequence in C converging to x. Take $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Because of the convergence of $\{x_n\}_{n=0}^{\infty}$ to x, there is a $M > N$ such that $|x_n - x| < 3^{-M}$, for all $n > M$. This implies that x und x_n lie in the same interval of C_n for all $n > M$, where

$$C_n = \{x \in [0,1] \mid d_i(x) \in \{0,2\}, \forall i \leq n\},$$

is the n-th approximation of the Cantor set C (see lecture). In particular, this shows that $d_i(x) = d_i(x_n)$ for any $i \leq M$. Consequently, we see that

$$|F(x_n) - F(x)| \leq \sum_{k=M+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^M} < \frac{1}{2^N} < \varepsilon,$$

which implies the continuity of F. □

(c) Show that F is surjective.

Solution: Let $y \in [0,1]$ be any element. The expansion of y in the basis 2 is assumed to be $y = \sum_{k=1}^{\infty} b_k \cdot 2^{-k}$ with $b_k \in \{0,1\}$. Define $a_k := 2b_k$ for all $k \geq 1$. In this case, $x = \sum_{k=1}^{\infty} a_k \cdot 3^{-k}$ is by definition an element of C (because $a_k \in \{0,2\}$) and it holds

$$F(x) = F\left(\sum_{k=1}^{\infty} \frac{a_k}{3^k}\right) = \sum_{k=1}^{\infty} \frac{a_k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = y.$$

Therefore, F is surjective. □

(d) Conclude that C is uncountable.

Solution: F is a continuous map, which sends C surjectively onto $[0,1]$. As $[0,1]$ is uncountable, the set C has to be uncountable as well. □

Exercise 5.3. ♣

Which of the following statements are true?

(a) There is a subset $A \subset \mathbb{R}$ which is not Lebesgue-measurable but such that the set $B := \{x \in A : x \text{ is irrational}\}$ is Lebesgue-measurable. ✗

(b) There exist two disjoint sets $A, B \subset \mathbb{R}^n$ which are not $\mathcal{L}^n$-measurable but whose union is $\mathcal{L}^n$-measurable. ✓

(c) If the boundary of $\Omega \subset \mathbb{R}^n$ has $\mathcal{L}^n$-measure zero, then Ω is $\mathcal{L}^n$-measurable. ✓

(d) Let $A \subset [0,1]$ be a set which is not $\mathcal{L}^1$-measurable. Then the set $B := \{(x,x) : x \in A\} \subset \mathbb{R}^2$ is not $\mathcal{L}^2$-measurable. ✗

Exercise 5.4.
In this exercise we want to prove that there is a one-to-one correspondence between the nondecreasing left-continuous[1] functions F on $\mathbb{R}$ with $F(0) = 0$ and the Borel measures on $\mathbb{R}$ that are finite on bounded Borel sets.

(a) Given any nondecreasing left-continuous function $F : \mathbb{R} \to \mathbb{R}$, show that the Lebesgue-Stieltjes measure Λ_F generated by F is the unique Borel measure on $\mathbb{R}$ that is equal to $F(b) - F(a)$ on $[a,b)$. Namely, for every other Borel measure μ on $\mathbb{R}$ such that $\mu([a,b)) = F(b) - F(a)$ we have that μ coincides with Λ_F on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

Solution: It was proven in the lectures that the Lebesgue-Stieltjes measure Λ_F is Borel and satisfies $\Lambda_F([a,b)) = F(b) - F(a)$ for $a, b \in \mathbb{R}, a < b$. Now let $\mathcal{A}$ be the algebra of finite disjoint unions of half-closed intervals in $\mathbb{R}$, possibly infinite on the left and on the right, namely

$$\mathcal{A} := \left\{ \bigcup_{i=1}^n [a_i, b_i) : -\infty \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq +\infty \right\},$$

with the understanding that if $a_1 = -\infty$ we exclude $-\infty$ from it (this is a variation of Exercise 1.1.10 (2) in the Lecture Notes). Since Λ_F is a Borel measure, it restricts to a pre-measure on $\mathcal{A}$. Define $F(-\infty)$ and $F(+\infty)$ as the corresponding limits, which exist in $\overline{\mathbb{R}}$ because F is monotone. Then it follows from the behaviour of measures with respect to unions that $\Lambda_F([a,b)) = F(a) - F(b)$ also for infinite a or b.

Given a Borel measure μ on $\mathbb{R}$ such that $\mu([a,b)) = F(b) - F(a)$ for finite a and b, we can also pass to the limit and obtain this equality for arbitrary $-\infty \leq a < b \leq +\infty$. Therefore

$$\mu\left(\bigcup_{i=1}^n [a_i, b_i)\right) = \sum_{i=1}^n \mu([a_i, b_i)) = \sum_{i=1}^n F(b_i) - F(a_i) = \sum_{i=1}^n \Lambda_F([a_i, b_i)) = \Lambda_F\left(\bigcup_{i=1}^n [a_i, b_i)\right).$$

Hence $\mu(A) = \Lambda_F(A)$ for every $A \in \mathcal{A}$. Since Λ_F is clearly σ-finite, by the uniqueness of the Carathéodory–Hahn extension (Theorem 1.2.21 in the Lecture Notes) μ coincides with Λ_F on $\mathcal{B}(\mathbb{R})$, as the σ-algebra generated by $\mathcal{A}$ is $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ (see Example 1.1.9 (2)). □

(b) Conversely, given any Borel measure μ on $\mathbb{R}$ that is finite on all bounded Borel sets, the function $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = \begin{cases} \mu([0,x)) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu([x,0)) & \text{if } x < 0 \end{cases}$$

is nondecreasing and left-continuous and μ coincides with Λ_F on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

[1] A function $F : \mathbb{R} \to \mathbb{R}$ is left-continuous if $\lim_{x \to a^-} F(x) = F(a)$ for every $a \in \mathbb{R}$.

Solution: Consider any $x < y$ in $\mathbb{R}$. If $x \leq 0 < y$, then clearly $F(x) \leq 0 \leq F(y)$. If $0 < x < y$, then $F(x) = \mu([0,x)) \leq \mu([0,x)) + \mu([x,y)) = \mu([0,y)) = F(y)$. Analogously we get $F(x) \leq F(y)$ for $x < y \leq 0$. Thus F is nondecreasing.

Now let $x_0 \in \mathbb{R}$, we assume $x_0 > 0$ (in the other cases the argument is analogous). Then, by the continuity from below of the measure (see Theorem 1.2.14), we have

$$F(x_0) = \mu([0,x_0)) = \lim_{x \to x_0^-} \mu([0,x)) = \lim_{x \to x_0^-} F(x),$$

which proves that F is left-continuous.

Now note that $\mu([a,b)) = F(b) - F(a)$ for all $a < b$. Therefore, by part (a), we have that $\mu = \Lambda_F$ on the Borel sets. □