Exercise 6.1.
Let $f : [0,1] \to \mathbb{R}$ be an α-Hölder continuous function, namely, there is a constant $L > 0$ such that $|f(x)-f(y)| \leq L|x-y|^\alpha$ for every $x, y \in [0,1]$, where $0 < \alpha \leq 1$ is a fixed number. Let $G = \{(x, f(x)) : x \in [0,1]\} \subset \mathbb{R}^2$ denote its graph.

(a) Show that $\mathcal{L}^2(G) = 0$.

Solution: We can show more generally that if $f : [0,1] \to \mathbb{R}$ is a continuous function, then its graph has Lebesgue measure zero. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_1, x_2 \in [0,1]$ satisfy $|x_1 - x_2| \leq \delta$, then $|f(x_1) - f(x_2)| \leq \varepsilon$.

Choose a subdivision $0 = a_0 < a_1 < \cdots < a_N = 1$ such that $a_i - a_{i-1} < \delta$ and consider the rectangles $R_i := [a_{i-1}, a_i] \times [f(a_i) - \varepsilon, f(a_i) + \varepsilon] \subset \mathbb{R}^2$, $i = 1, \ldots, N$. We claim that these rectangles cover the graph of f: given $x \in [0,1]$, say $x \in [a_{i-1}, a_i]$ for some i, since $|x - a_i| \leq |a_{i-1} - a_i| \leq \delta$, then $|f(x) - f(a_i)| \leq \varepsilon$ and thus $(x, f(x)) \in R_i$, which proves the claim.

Now, by the definition of the Lebesgue measure,

$$\mathcal{L}^2(G) \leq \sum_{i=1}^N \operatorname{vol}(R_i) = \sum_{i=1}^N (a_i - a_{i-1}) \cdot 2\varepsilon = 2\varepsilon \sum_{i=1}^N a_i - a_{i-1} = 2\varepsilon(1-0) = 2\varepsilon$$

and letting $\varepsilon \to 0$ it follows that $\mathcal{L}^2(G) = 0$.

(b) ★ Show moreover that $\mathcal{H}^s(G) = 0$ for every $s > 2 - \alpha$.

Solution: Let $N > 0$ be a natural number and consider the set of squares

$$\mathcal{Q}_N := \{Q_{i,j} : i = 0, \ldots, N-1,\ j \in \mathbb{Z}\}$$

where

$$Q_{i,j} := \left[\frac{i}{N}, \frac{i+1}{N}\right] \times \left[\frac{j}{N}, \frac{j+1}{N}\right].$$

We claim that, if N is large enough, then for each i there are at most $2LN^{1-\alpha}$ squares $Q_{i,j}$ that intersect G. Indeed, fix one j_0 such that $Q_{i,j_0} \cap G \neq \varnothing$ and consider any other j_1 such that $Q_{i,j_1} \cap G \neq \varnothing$. Then there exist $x_0, x_1 \in [i/N, (i+1)/N]$ such that $j_0/N \leq f(x_0) \leq (j_0+1)/N$ and $j_1/N \leq f(x_1) \leq (j_1+1)/N$. But since $|f(x_0) - f(x_1)| \leq L|x_0 - x_1|^\alpha \leq L\left(\frac{1}{N}\right)^\alpha$, it must hold

$$|j_0 - j_1| = N \cdot \left|\frac{j_0}{N} - \frac{j_1}{N}\right| \leq N \cdot \left(\left|\frac{j_0}{N} - f(x_0)\right| + |f(x_0) - f(x_1)| + \left|f(x_1) - \frac{j_1}{N}\right|\right) \leq N \cdot \left(\frac{2}{N} + \frac{L}{N^\alpha}\right),$$

so that $|j_0 - j_1| \leq 2 + LN^{1-\alpha} \leq 2LN^{1-\alpha}$ if N is large enough. As a consequece, in total there are at most $N \cdot 2LN^{1-\alpha} = 2LN^{2-\alpha}$ squares $Q \in \mathcal{Q}_N$ that intersect G.

To work with the definition of Hausdorff measure given in the lectures we need to work with balls instead of squares. Thus, for each pair of indices (i,j) as before, we define $B_{i,j}$ to be the ball with the same center as $Q_{i,j}$ and with radius $1/N$, in such a way that $Q_{i,j} \subset B_{i,j}$. Now consider the collection

$$\mathcal{B}_N := \{B_{i,j} : Q_{i,j} \cap G \neq \varnothing\},$$

which is clearly a covering of G with at most $2LN^{2-\alpha}$ balls.

Now fix $s > 2 - \alpha$ and let $\delta > 0$. It follows from the definition of $\mathcal{H}^s_\delta$ that, if $N > 1/\delta$, then

$$\mathcal{H}^s_\delta(G) \leq \left(\frac{1}{N}\right)^s \cdot \#(\mathcal{B}_N) \leq 2LN^{2-\alpha-s}.$$

Letting $N \to \infty$ we find that $\mathcal{H}^s_\delta(G) = 0$, and finally letting $\delta \to 0$ we obtain that $\mathcal{H}^s(G) = 0$. $\square$

Exercise 6.2.
Show that the graph of the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

has Lebesgue measure zero in $\mathbb{R}^2$.

Solution: The function g is not continuous, but for each $a > 0$, $g|_{[a,1]}$ is clearly continuous (and even smooth, so α-Hölder for every α). Therefore, by Exercise 6.1(a), for each $j \in \mathbb{Z}^+$, $\mathcal{L}^2\left(\operatorname{graph} g|_{[j^{-1},1]}\right) = 0$, so by the continuity property of a measure for increasing sequences of sets,

$$\mathcal{L}^2(\operatorname{graph} g) = \mathcal{L}^2\left(\bigcup_{j \in \mathbb{Z}^+} \operatorname{graph} g|_{[j^{-1},1]}\right) = \lim_{j \to \infty} \mathcal{L}^2\left(\operatorname{graph} g|_{[j^{-1},1]}\right) = 0. \qquad \square$$

Exercise 6.3.
For $s \geq 0$ and $\emptyset \neq A \subset \mathbb{R}^n$, we define

$$\mathcal{H}^s_\infty(A) := \inf\left\{\sum_{k \in I} r_k^s : A \subset \bigcup_{k \in I} B(x_k, r_k),\ r_k > 0\right\},$$

where the set of indices I is at most countable. One can check that $\mathcal{H}^s_\infty$ is a measure. Prove that $\mathcal{H}^{1/2}_\infty$ is not Borel on $\mathbb{R}$.

Remark. Note that the definition of $\mathcal{H}^s_\infty$ coincides with Definition 1.8.1 in the Lecture Notes for $\delta = \infty$.

Solution: We show that the interval $[0,1]$ is not $\mathcal{H}^{1/2}_\infty$-measurable, from which follows that $\mathcal{H}^{1/2}_\infty$ is not Borel on $\mathbb{R}$.

First let us prove that $\mathcal{H}^{1/2}_\infty([a,b]) = (\frac{b-a}{2})^{1/2}$ for all $a < b$. Note that the interval $B(\frac{a+b}{2}, \frac{b-a}{2} + \varepsilon)$ covers $[a,b]$ for all $\varepsilon > 0$. Therefore we have that $\mathcal{H}^{1/2}_\infty([a,b]) \leq \left(\frac{b-a}{2} + \varepsilon\right)^{1/2}$, which implies that $\mathcal{H}^{1/2}_\infty([a,b]) \leq \left(\frac{b-a}{2}\right)^{1/2}$ for arbitrariness of ε. On the other hand, given any finite or countable cover $\{B(x_k, r_k)\}_{k \in I}$ of $[a,b]$, the total length of the intervals of the covering should be at least $b-a$, namely $\sum_{k \in I} 2r_k \geq b - a$. Hence, using that $(\sum_{k \in I} r_k^{1/2})^2 \geq \sum_{k \in I} r_k$, we get

$$\sum_{k \in I} r_k^{1/2} \geq \left(\sum_{k \in I} r_k\right)^{1/2} \geq \left(\frac{b-a}{2}\right)^{1/2}.$$

Therefore we obtain that $\mathcal{H}^{1/2}_\infty([a,b]) = (\frac{b-a}{2})^{1/2}$ for all $a < b$. The same proof (with the same result) works for half-closed and open intervals.

As a result, we get

$$\mathcal{H}^{1/2}_\infty([0,2]) = 1 \neq 2^{3/2} = \mathcal{H}^{1/2}_\infty([0,1]) + \mathcal{H}^{1/2}_\infty((1,2]),$$

which proves that $[0,1]$ is not $\mathcal{H}^{1/2}_\infty$-measurable. □

Exercise 6.4. ♣

Recall that a measure is called Radon if it is Borel regular and finite on compact sets.

Which of the following statements are true?

(a) The Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ is a Radon measure. ✓

(b) For any nondecreasing and left-continuous function $F : \mathbb{R} \to \mathbb{R}$, the Lebesgue–Stieltjes measure Λ_F is a Radon measure on $\mathbb{R}$. ✓

(c) For any $s > 0$, the Hausdorff measure $\mathcal{H}^s$ is a Radon measure on $\mathbb{R}^n$. ✗

(d) The Dirac measure δ_0 is a Radon measure on $\mathbb{R}$. ✓

(e) For every set $A \subset \mathbb{R}^n$, it holds $\mathcal{H}^{n+1}(A) = 0$. ✓

Exercise 6.5.

Consider the continuous function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

and its graph

$$A := \{(x, f(x)) \mid x \in [0,1]\} \subset \mathbb{R}^2.$$

(a) ★ Show that $\mathcal{H}^1(A) = \infty$.

Hint: use the σ-additivity of $\mathcal{H}^1$ on Borel sets to decompose the curve A into pieces which "look like" straight lines and try to compare their $\mathcal{H}^1$-measure with their length.

Solution: Let $a_n = 1/(\pi n)$ for each $n \in \mathbb{Z}^+$. By the σ-additivity of $\mathcal{H}^1$ (and the fact that Borel sets are measurable), we can write

$$\mathcal{H}^1(A) \geq \sum_{n\geq 1} \mathcal{H}^1\left(\text{graph}\left(f|_{[a_{n+1},a_n]}\right)\right).$$

Observe that $\text{dist}((a_n, f(a_n)), (a_{n+1}, f(a_{n+1}))) \geq |f(a_n) - f(a_{n+1})| = |f(a_n)| + |f(a_{n+1})| = |a_n| + |a_{n+1}| \geq 1/(\pi n)$. Thus, since the harmonic series diverges, it is enough to prove that for a smooth connected curve γ joining two points p and q in $\mathbb{R}^2$, $\mathcal{H}^1(\gamma) \geq c \, \text{dist}(p,q)$ for some constant $c > 0$ (we will actually show this with $c = 1/2$, which is sharp).

To see that, consider a covering $\mathscr{B}$ of γ by balls and extract a finite subcovering $B(x_1, r_1), \ldots, B(x_N, r_N)$; moreover we may assume that all these balls intersect γ. Let $i_1 \in \{1, \ldots, N\}$ be such that $p \in B(x_{i_1}, r_{i_1})$; in particular $B(x_{i_1}, r_{i_1}) \subset B(p, 2r_{i_1})$. Since γ is connected, the open sets $B(p, 2r_{i_1})$ and $\bigcup_{j \neq i_1} B(x_j, r_j)$ cannot disconnect γ and therefore they must have a point in common inside γ. Thus there exists an index $i_2 \neq i_1$ such that $B(p, 2r_{i_1}) \cap B(x_{i_2}, r_{i_2}) \neq \varnothing$. Now it is easy to see that the enlarged ball $B(p, 2r_{i_1} + 2r_{i_2})$ contains $B(x_{i_2}, r_{i_2})$ as well, so we may iterate the argument and consider the covering of γ by $B(p, 2r_{i_1} + 2r_{i_2})$ and $\bigcup_{j \neq i_1, i_2} B(x_j, r_j)$ to find a third index i_3 with $B(p, 2r_{i_1} + 2r_{i_2}) \cap B(x_{i_3}, r_{i_3}) \neq \varnothing$ and deduce $B(x_{i_3}, r_{i_3}) \subset B(p, 2r_{i_1} + 2r_{i_2} + 2r_{i_3})$.

By repeating this argument we conclude that all the original balls are contained in $B(p, 2r_{i_1} + \cdots + 2r_{i_N}) = B(p, 2r_1 + \cdots + 2r_N)$; in particular, q is also contained there, and it follows that $\operatorname{dist}(p, q) \leq 2(r_1 + \cdots + r_N) \leq 2 \sum_{B \in \mathscr{B}} \operatorname{radius}(B)$. Fix $\delta > 0$ and take the infimum over such collections of balls with radius at most δ to deduce that $\mathcal{H}^s_\delta(\gamma) \geq \operatorname{dist}(p, q)/2$, so the same holds when we take the limit $\delta \searrow 0$. □

Remark: It is also possible to show this by considering the projection of the graph of f restricted to $[a_{n+1}, a_n]$ onto the Y-axis, which (as one can easily show) does not increase Hausdorff measures, and conclude by comparing the $\mathcal{H}^1$ and $\mathcal{L}^1$ measures on the real line.

(b) Show that $\mathcal{H}^s(A) = 0$ if $s > 1$.

Solution: We use the same trick as in Exercise 6.2: we can write

$$A = \operatorname{graph}(f) = \bigcup_{j \in \mathbb{Z}^+} \operatorname{graph}\left(f|_{[j^{-1}, 1]}\right)$$

and pass to the limit once we show that $\operatorname{graph}\left(f|_{[a,1]}\right)$ has $\mathcal{H}^1$ measure zero for each $a > 0$. But this is a consequence of Exercise 6.1(b): $f|_{[a,1]}$ is smooth and in particular Lipschitz (that is, α-Hölder with $\alpha = 1$), hence the condition $s > 2 - \alpha = 1$ holds. □

Remark. A more explicit solution can be given by covering A by small balls and using directly the definition of the Hausdorff measure.

(c) Conclude that $\dim_{\mathcal{H}}(A) = 1$.

Solution: The fact that $\mathcal{H}^1(A) = \infty$ implies that $\dim_{\mathcal{H}}(A) \geq 1$, and the fact that $\mathcal{H}^s(A) = 0$ for any $s > 1$ implies that $\dim_{\mathcal{H}}(A) \leq s$ for any $s > 1$. Therefore it must be $\dim_{\mathcal{H}}(A) = 1$. □