**Exercise 7.1.** ♣
Let $\mu$ be a measure on $\mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ a $\mu$-measurable set. Which of the following statements are true?

(a) If $f : \Omega \to \mathbb{R}$ is $\mu$-measurable and $g : \mathbb{R} \to \mathbb{R}$ is a Borel function[1], then $g \circ f$ is $\mu$-measurable. ✓

(b) Let $f : [0,1] \to \mathbb{R}$ and suppose that for every $c \in \mathbb{R}$, the set $\{x \in [0,1] : f(x) = c\}$ is $\mathcal{L}^1$-measurable. Then $f$ is $\mathcal{L}^1$-measurable. ✗

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is continuous $\mathcal{L}^1$-almost everywhere. Then $f$ is $\mathcal{L}^1$-measurable. ✓

(d) Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of $\mu$-measurable functions $f_k : \mathbb{R}^n \to \mathbb{R}$. Then the set

$$E := \{x \in \mathbb{R}^n : \lim_{k\to\infty} f_k(x) \text{ exists and is finite}\}$$

is $\mu$-measurable. ✓

**Exercise 7.2.**
Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Show that the following statements are equivalent.

(i) $f^{-1}(U)$ is $\mu$-measurable for every open set $U \subset \mathbb{R}$.

(ii) $f^{-1}(B)$ is $\mu$-measurable for every Borel set $B \subset \mathbb{R}$.

(iii) $f^{-1}((-\infty, a))$ is $\mu$-measurable for every $a \in \mathbb{R}$.

**Solution:** (i) $\Leftrightarrow$ (ii): As all open subsets $U \subset \mathbb{R}$ are Borel, it is obvious that (ii) $\Rightarrow$ (i). On the other hand, the following collection $\{B \subset \mathbb{R} \mid f^{-1}(B)\ \mu\text{-measurable}\}$ is a $\sigma$-algebra, see Exercise 1.4 (b). If this $\sigma$-algebra contains all open subsets, then it contains all Borel sets, which proves that (i) $\Rightarrow$ (ii).

(ii) $\Leftrightarrow$ (iii): Once more, it is clear that (ii) $\Rightarrow$ (iii), because the intervals $(-\infty, a)$ are Borel sets for all $a \in \mathbb{R}$. On the other hand, we know that $((-\infty, a))_{a\in\mathbb{R}}$ generates the Borel $\sigma$-algebra, which proves the other implication. □

**Exercise 7.3.**
Let $(X, \mu, \Sigma)$ be a measure space and $f, g : X \to \mathbb{R}$ two measurable functions on $X$. Show that the sets $\{x \mid f(x) = g(x)\}$ and $\{x \mid f(x) < g(x)\}$ are measurable.

[1] A function $g : \mathbb{R} \to \mathbb{R}$ is Borel iff $g^{-1}(U)$ is a Borel set for every open set $U \subseteq \mathbb{R}$.

**Solution:** Since $f$ and $g$ are measurable, then $h := f - g$ is measurable as well. As a result, we know that

$$\{x \mid f(x) = g(x)\} = h^{-1}(\{0\})$$

is measurable and the same holds for

$$\{x \mid f(x) < g(x)\} = h^{-1}((-\infty, 0)).$$

□

**Exercise 7.4.** ★

In this exercise, we construct a set which is Lebesgue measurable but not Borel, and use the construction to give an example of a continuous $G : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $H : \mathbb{R} \to \mathbb{R}$ such that $H \circ G$ is not Lebesgue measurable.

(a) Let $h : [0, 1] \to [0, 1]$ be the Cantor function, which is the unique monotonically increasing extension of the function $F : C \to [0, 1]$ seen in Exercise 5.2, where $C \subset [0, 1]$ is the Cantor set. Define $g : [0, 1] \to [0, 2]$ by $g(x) := h(x) + x$. Show that $g$ is strictly monotone and a homeomorphism.

**Solution:** Strict monotonicity is a direct consequence of $h$ being monotonically increasing and $x \mapsto x$ being strictly increasing. We just have to check whether $g^{-1}$ is continuous. As $[0, 1]$ is compact, the image under $g$ of each closed subset is a compact subset of $[0, 2]$, hence closed. By bijectivity, this implies that $g$ is open and thus a homeomorphism. □

(b) Show that $\mathcal{L}^1(g(C)) = 1$.

**Hint:** Use the natural decomposition of $[0, 1] \setminus C$ to deduce the result.

**Solution:** Observe that

$$[0, 1] \setminus C = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k},$$

where $I_{n,k}$ is the $k$-th interval removed in the $n$-th step of the construction of $C$ and has length $3^{-n}$. Hence we have

$$\mathcal{L}^1([0, 2] \setminus g(C)) = \mathcal{L}^1(g([0, 1] \setminus C)) = \mathcal{L}^1\left(g\left(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}\right)\right) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \mathcal{L}^1(g(I_{n,k})).$$

To conclude, notice that $h$ is constant on each $I_{n,k}$. Therefore, we easily deduce $\mathcal{L}^1(g(I_{n,k})) = \mathcal{L}^1(I_{n,k}) = 3^{-n}$. Inserting this into the sequence of equations above, we conclude

$$\mathcal{L}^1([0, 2] \setminus g(C)) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1,$$

which implies

$$1 + \mathcal{L}^1(g(C)) = \mathcal{L}^1([0, 2] \setminus g(C)) + \mathcal{L}^1(g(C)) = \mathcal{L}^1([0, 2]) = 2,$$

and this implies the desired result. □

(c) Use Exercise 4.3 (a) to find a non-measurable subset $E \subset g(C)$ and define $A := g^{-1}(E)$. Show that $A$ is a Lebesgue zero set and thus Lebesgue measurable.

**Solution:** Observe that $A = g^{-1}(E) \subset g^{-1}(g(C)) = C$. As $C$ is a Lebesgue zero set, so is $A$, and consequently $A$ is Lebesgue measurable. □

(d) Show that $A$ is not a Borel set.

**Hint:** Otherwise, the preimage of $A$ with respect to continuous maps would necessarily be Borel and thus Lebesgue measurable as well.

**Solution:** Assume $A$ were Borel. Then, due to $g^{-1}$ being continuous by the first part of this exercise, we know

$$(g^{-1})^{-1}(A) = g(A) = g(g^{-1}(E)) = E \text{ is a Borel set.}$$

However, $E$ is not Lebesgue measurable and hence not Borel, contradicting the conclusion. Therefore, $A$ is not Borel. □

(e) Find appropriate $H, G$ as outlined above such that $H \circ G$ is not Lebesgue measurable, using the sets and functions introduced in the previous subtasks.

**Solution:** Let us take $H = \chi_A$ and $G = g^{-1}$, where $g$ and $A$ are as previously introduced. We have thus seen that $H$ is Lebesgue measurable and $G$ is continuous. Assume that $H \circ G$ is Lebesgue measurable. Note that $\{1\}$ is a closed subset and thus, $(H \circ G)^{-1}(\{1\})$ would be Lebesgue measurable. But

$$(H \circ G)^{-1}(\{1\}) = G^{-1}\big(H^{-1}(\{1\})\big) = G^{-1}(A) = g(A) = E,$$

which is not Lebesgue measurable, contradiction. □

**Exercise 7.5.**
Let $\mu$ be a Borel measure on $\mathbb{R}$. Show that every monotone function $f\colon [a,b] \to \mathbb{R}$ is $\mu$-measurable.

**Solution:** As $f^{-1}((-\infty, c))$ is either an interval in $[a,b]$ or the empty set (when $c \notin f([a,b])$), $f$ is $\mu$-measurable according to Exercise 7.2. □