Exercise 8.1.
Let μ be a measure on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ a μ-measurable set and $f : \Omega \to [0, \infty]$ a μ-measurable function. Consider the sets $A_j \subseteq \Omega$ from Theorem 2.2.6 of the Lecture Notes, defined so that the sequence of functions

$$f_k = \sum_{j=1}^{k} \frac{1}{j} \chi_{A_j}$$

converges pointwise to f. Show that if f is bounded, then f_k converge uniformly to f, that is,

$$\sup_{x \in \Omega} |f(x) - f_k(x)| \longrightarrow 0 \text{ as } k \to \infty.$$

Solution: Suppose that $f(x) \leq M$ for every $x \in \Omega$ and let $k_0 > 2$ be large enough so that

$$\sum_{j=1}^{k_0} \frac{1}{j} > M.$$

Given $x \in \Omega$ and $k \geq k_0$, let j be the largest integer $\leq k$ such that $x \notin A_j$. Notice that such j must exist because otherwise we would have $f(x) \geq \sum_{j=1}^{k} \frac{1}{j} > M$. In this case $f_j(x) = f_{j-1}(x)$ and moreover, by definition of A_j,

$$f(x) < f_{j-1}(x) + \frac{1}{j} = f_j(x) + \frac{1}{j}, \tag{1}$$

but $x \in A_\ell$ for every $j < \ell \leq k$, which implies that

$$f_j(x) + \sum_{\ell=j+1}^{k} \frac{1}{\ell} \leq f(x). \tag{2}$$

Putting together (1) and (2) we get that

$$\sum_{\ell=j+1}^{k} \frac{1}{\ell} < \frac{1}{j}.$$

It is easy to check that such inequality cannot hold if $k - j \geq 3$, for example because of the easy inequality

$$\frac{1}{j} = \frac{1}{2j} + \frac{1}{3j} + \frac{1}{6j} < \frac{1}{j+1} + \frac{1}{j+2} + \frac{1}{j+3}$$

which is true for $j \geq 1$. Thus $j \geq k - 2$. Now (1) and the monotonicity of f_k imply

$$0 \leq f(x) - f_k(x) \leq f(x) - f_j(x) < \frac{1}{j} \leq \frac{1}{k-2},$$

from which the uniform convergence follows. □

Exercise 8.2.
Let $f : \Omega \to \overline{\mathbb{R}}$ be μ-measurable. Prove that there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of simple functions $f_k : \Omega \to \mathbb{R}$ satisfying $|f_k(x)| \leq |f_{k+1}(x)|$ and $\lim_{k\to\infty} f_k(x) = f(x)$ for every $x \in \Omega$.

In particular, this shows that $|f_k(x)| \leq |f(x)|$ for each $x \in \Omega$ and $k \geq 1$.

Solution: Write $f = f^+ - f^-$, where $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$. The construction of Theorem 2.2.6 of the Lecture Notes gives two sequences of simple functions s_k^+, s_k^- such that $s_k^\pm \nearrow f^\pm$ monotonically and pointwise. Thus it follows that $f_k := s_k^+ - s_k^-$ converge pointwise to $f = f^+ - f^-$ and $|f_k| = s_k^+ + s_k^- \leq s_{k+1}^+ + s_{k+1}^- = |f_{k+1}|$ as desired. □

Exercise 8.3. ♣
Which of the following assertions are true?

(a) Given a sequence $\{f_k\}$ of Lebesgue-measurable functions on $\mathbb{R}$ converging to 0 in measure, there is a subset $A \subset \mathbb{R}$ of positive measure and a subsequence which converges uniformly on A. ✓

(b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is Lebesgue-measurable. ✓

(c) There exists a function $f : \Omega \to [0, \infty)$ such that f is not measurable but $\sqrt{f}$ is measurable. ✗

(d) The sequence $f_n(x) = e^{-n(1-\sin x)}$ converges in measure to the function $f \equiv 0$ on any bounded interval $[a, b] \subset \mathbb{R}$. ✓

Exercise 8.4.
Let $f_k \colon \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-measurable functions, for $k \in \mathbb{N}$. Assume that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \mid |f_k(x) - f_{k+1}(x)| > 2^{-k}\}) < 2^{-k}$$

for all $k \in \mathbb{N}$. Show that the limit $\lim\limits_{k\to\infty} f_k(x)$ exists almost everywhere.

Solution: Define $A_k = \{x \in \mathbb{R}^n \mid |f_k(x) - f_{k+1}(x)| \leq 2^{-k}\}$. By assumption, we have $\mathcal{L}^n(A_k^c) < 2^{-k}$. Let $B_l := \cap_{k\geq l} A_k$, then $B_{l+1} \supset B_l$ and equivalently $B_{l+1}^c \subset B_l^c$. Since

$$\mathcal{L}^n(B_l^c) \leq \sum_{k\geq l} \mathcal{L}^n(A_k^c) < \sum_{k\geq l} 2^{-k} = 2^{-l+1}$$

(in particular $\mathcal{L}^n(B_1^c) \leq 1$), it follows

$$\mathcal{L}^n\left(\Big(\bigcup_{l\in\mathbb{N}} B_l\Big)^c\right) = \mathcal{L}^n\Big(\bigcap_{l\in\mathbb{N}} B_l^c\Big) = \lim_{l\to\infty} \mathcal{L}^n(B_l^c) \leq \lim_{l\to\infty} 2^{-l+1} = 0. \tag{3}$$

For $x \in B_l$ and $l < m < n \in \mathbb{N}$, it holds by the triangle inequality:

$$|f_m(x) - f_n(x)| \leq \sum_{k=m}^{n-1} |f_k(x) - f_{k+1}(x)| \leq \sum_{k=m}^{n-1} 2^{-k} \leq 2^{-m+1}.$$

Hence, for every $x \in B_l$, $f_k(x)$ is a Cauchy sequence and consequently, the limit $\lim_{k\to\infty} f_k(x)$ exists. Because of (3), $\bigcup_{l\in\mathbb{N}} B_l$ is almost everywhere and therefore $\lim_{k\to\infty} f_k(x)$ exists for almost all $x \in \mathbb{R}^n$. $\square$

Exercise 8.5.

Let μ be a measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ be μ-measurable. Let $f \colon \Omega \to \overline{\mathbb{R}}$ be a finite, μ-measurable function, and $(f_k)_{k\in\mathbb{N}}$ a sequence of μ-measurable functions $f_k \colon \Omega \to \overline{\mathbb{R}}$.

(a) Suppose that every subsequence $(f_{k_j})_{j\in\mathbb{N}}$ contains a subsequence that converges to f in measure. Show that the whole sequence $(f_k)_{k\in\mathbb{N}}$ converges to f in measure.

Solution: Suppose the opposite was true. Then there exist $\varepsilon > 0$, $\delta > 0$ and a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$, such that $\mu(\{x \mid |f(x) - f_{k_j}(x)| > \varepsilon\}) > \delta$ for all $j \in \mathbb{N}$. This subsequence cannot contain another subsequence converging in measure μ. Therefore, $\{f_k\}_{k\in\mathbb{N}}$ converges in measure. $\square$

(b) Show that the analogous statement from (a) is not true if we replace "convergence in measure" by "convergence pointwise almost everywhere". Namely, show that there exists a sequence (f_k) and a function f such that every subsequence of (f_k) has a further subsequence that converges a.e. to f, but the whole (f_k) does not converge a.e. to any function.

Solution: A counterexample is provided by the sequence $f_k : [0, 1) \to \mathbb{R}$ with $f_k = \chi_{[k/2^n - 1, (k+1)/2^n - 1))}$ for $2^n \leq k < 2^{n+1}$. For any $x \in [0, 1)$, the sequence $(f_k(x))_{k\in\mathbb{N}}$ is not convergent.

Claim: Every subsequence of $\{f_k\}_{k\in\mathbb{N}}$ possesses an $\mathcal{L}^1$-almost everywhere convergent subsequence.

Proof: Let $\{g_j\}_{j\in\mathbb{N}} = \{f_{k_j}\}_{j\in\mathbb{N}}$ be a subsequence of $\{f_k\}_{k\in\mathbb{N}}$. We inductively construct a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ satisfying the following conditions:

1. $\mathcal{L}^1(I_n) = 2^{-n}$;
2. For any $n \in \mathbb{N}$, there is a subsequence $\{g_j^{(n)}\}_{j\in\mathbb{N}}$ of $\{g_j\}_{j\in\mathbb{N}}$ such that $\operatorname{supp}(g_j^{(n)}) \subset I_n$;
3. $\{g_j^{(n+1)}\}_{j\in\mathbb{N}}$ is a subsequence of $\{g_j^{(n)}\}_{j\in\mathbb{N}}$.

For $n = 1$, we choose the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. For any g_j we either have $\operatorname{supp} g_j \subset [0, \frac{1}{2})$ or $\operatorname{supp} g_j \subset [\frac{1}{2}, 1)$. As a result, at least one of the intervals contains infinitely many of the supports of g_j. We denote this interval by I_1. The g_j's with support in I_1 form the subsequence $\{g_j^{(1)}\}_{j\in\mathbb{N}}$.

Let $\{g_j^{(n)}\}_{j\in\mathbb{N}}$ be a sequence with the properties above. We define the intervals $K_l = [l \cdot 2^{-(n+1)}, (l + 1) \cdot 2^{-(n+1)})$ for $l = 0, \ldots, 2^{n+1} - 1$. For all $g_j^{(n)}$ with j sufficiently large, there is $l = l(j)$ such that $\operatorname{supp}(g_j^{(n)}) \subset K_l$. As a result, at least one of the K_l, which we denote by I_{n+1}, contains the support of infinitely many $g_j^{(n)}$. These $g_j^{(n)}$ form the subsequence $\{g_j^{(n+1)}\}_{j\in\mathbb{N}}$.

Using the construction above, we take a diagonal sequence $h_m := g_m^{(m)}$. Note that $\{h_m\}_{m\in\mathbb{N}}$ is a subsequence of $\{f_{k_j}\}_{j\in\mathbb{N}}$. Let $N := \bigcap_{n\in\mathbb{N}} I_n$. Because of upper continuity of the measure, we have $\mathcal{L}^1(N) = \lim_{n\to\infty} \mathcal{L}^1(I_n) = 0$.

Now let $x \notin N$. Then there is a $n = n(x)$, such that $x \notin I_{n(x)}$. Consequently, $h_m(x) = 0$ for all $m > n(x)$. So h_m converges pointwise $\mathcal{L}^1$-almost everywhere to zero. □

Exercise 8.6. ★

Counterexample to $\varepsilon = 0$ in Lusin's Theorem: Find an example of a $\mathcal{L}^1$-measurable function $f : [0,1] \to \mathbb{R}$ such that for every $\mathcal{L}^1$-measurable set $M \subset [0,1]$ with $\mathcal{L}^1(M) = 1$, the restriction $f|_M : M \to \mathbb{R}$ is discontinuous in all but finitely many points of M.

Hint: You may use that there exists a Lebesgue measurable subset $A \subset [0,1]$ such that

$$\mathcal{L}^1(U \cap A) \cdot \mathcal{L}^1(U \cap A^c) > 0$$

for all nonempty open subsets $U \subset [0,1]$. Such a set A can be constructed using the fat Cantor set (see Exercise 1.6.2 in the lecture notes).

Solution: Let $f = \chi_A$ where $A \subset [0,1]$ is as in the hint. This set will be constructed in a forthcoming exercise. Moreover, let $M \subset [0,1]$ as described above. We show that $f|_M$ is discontinuous in every point except for $\{0,1\}$. Let $x \in M \setminus \{0,1\}$ and choose sequences $a_n \le x \le b_n$ that converge monotonically to x. Observe that, for all $I_n := (a_n, b_n)$, it holds

$$\mathcal{L}^1(I_n \cap A) \cdot \mathcal{L}^1(I_n \cap A^c) > 0.$$

Using that $\mathcal{L}^1([0,1] \setminus M) = 0$ as well as Caratheodory's characterisation of measurability, we get $\mathcal{L}^1(I_n \cap A) = \mathcal{L}^1(I_n \cap A \cap M) + \mathcal{L}^1((I_n \cap A) \setminus M) = \mathcal{L}^1(I_n \cap A \cap M)$ and analogously $\mathcal{L}^1(I_n \cap A^c) = \mathcal{L}^1(I_n \cap A^c \cap M)$. Therefore, the previous inequality can be read as

$$\mathcal{L}^1(I_n \cap A \cap M) \cdot \mathcal{L}^1(I_n \cap A^c \cap M) > 0.$$

This implies that there exists $x_n, y_n \in I_n$ such that

$$x_n \in I_n \cap A \cap M, \quad y_n \in I_n \cap A^c \cap M,$$

therefore $f(x_n) = 1, f(y_n) = 0$. Observe that $x_n \to x$ and similarly $y_n \to x$. This provides the desired contradiction to continuity. □

Exercise 8.7.

Counterexample to $\delta = 0$ in Egoroff's Theorem: Find an example of a sequence of $\mathcal{L}^1$-measurable functions $f_k : [0,1] \to \overline{\mathbb{R}}$ that converges pointwise almost everywhere to a $\mathcal{L}^1$-measurable ($\mathcal{L}^1$-almost everywhere finite) function $f : [0,1] \to \overline{\mathbb{R}}$, but for every set $F \subseteq [0,1]$ with $\mathcal{L}^1(F) = \mathcal{L}^1([0,1])$ the convergence on F is not uniform.

Solution: Let $f_k(x) = x^k$ for $x \in [0,1]$, which converges pointwise to $f \equiv 0$ on $[0,1)$ and $f(1) = 1$. Let $F \subset [0,1]$ be any set of full measure, and suppose that f_k converges uniformly in F. Therefore, there exists some $K \in \mathbb{N}$ such that $|f_k(x) - f(x)| < \frac{1}{2}$ for $k \geq K$. In particular, for every $x \in F \cap [0,1)$ it must hold that $x^K < \frac{1}{2}$, so that $x < 1/\sqrt[K]{2}$. This implies that F is disjoint with the interval $(1/\sqrt[K]{2}, 1)$, and thus cannot have full measure. □