Exercise 9.1. ♣
Which of the following statements are true?

4 correct answers are enough for the bonus.

(a) Let $\{f_k\}$ be a sequence of non-negative measurable functions on $\mathbb{R}$ such that $f_k \to f$ almost everywhere. Then $\lim_{k\to\infty} \int_{\mathbb{R}} f_k \, d\mathcal{L}^1$ exists and

$$\int_{\mathbb{R}} f \, d\mathcal{L}^1 \leq \lim_{k\to\infty} \int_{\mathbb{R}} f_k \, d\mathcal{L}^1.$$

Solution: ✗ The limit may not exist, for example, set $f_k = \chi_{[k,k+1]}$ for k even and $f_k \equiv 0$ for k odd; then this sequence converges pointwise to zero but the values of the integrals oscillate.

(b) Let $f : [0,1] \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Then for each nonnegative integer k, $x^k f(x)$ is $\mathcal{L}^1$-summable in $[0,1]$. ✓

(c) Let $f : (0,+\infty) \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Then $\lim_{x\to+\infty} |f(x)| = 0$. ✗

(d) Let $f : (0,+\infty) \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Then there exists a sequence $x_n \to \infty$ such that $\lim_{n\to\infty} x_n f(x_n) = 0$.

Solution: ✓ Assume the opposite: this means that there is some $\varepsilon > 0$ and $K \in \mathbb{R}$ such that $x|f(x)| \geq \varepsilon$ for all $x \geq K$. Therefore $f(x) \geq \varepsilon/x$ on $[K,+\infty)$, which is not a summable function, giving a contradiction.

(e) There exists a sequence $\{f_n\}$ of $\mathcal{L}^1$-summable functions on $(0,\infty)$ such that $|f_n(x)| \leq 1$ for all x and all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and $\lim_{n\to\infty} \int_{(0,\infty)} f_n \, d\mathcal{L}^1 = 1$. ✓

(f) There exists a sequence $\{f_n\}$ of $\mathcal{L}^1$-integrable functions on $[0,1]$ such that $f_n \to 0$ pointwise and yet $\int_{[0,1]} f_n \, d\mathcal{L}^1 \to +\infty$. ✓

Exercise 9.2.

(a) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of μ-measurable functions on a μ-measurable set $\Omega \subset \mathbb{R}^n$. Show that the series $\sum_{k=1}^{\infty} f_k(x)$ converges μ-almost everywhere, if

$$\sum_{k=1}^{\infty} \int_{\Omega} |f_k| d\mu < \infty.$$

Solution: Let us define

$$g_k := \sum_{j=1}^{k} |f_j|$$

and it obviously holds $g_k \leq g_{k+1}$ for all $k \geq 1$. Using monotone convergence of integrals, we see

$$\int_\Omega \sum_{j=1}^{\infty} |f_j| \, d\mu = \int_\Omega \lim_{k\to\infty} g_k \, d\mu = \lim_{k\to\infty} \int_\Omega g_k \, d\mu = \lim_{k\to\infty} \int_\Omega \sum_{j=1}^{k} |f_j| \, d\mu$$
$$= \lim_{k\to\infty} \sum_{j=1}^{k} \int_\Omega |f_j| \, d\mu = \sum_{j=1}^{\infty} \int_\Omega |f_j| \, d\mu.$$

Since $\int_\Omega \sum_{j=1}^\infty |f_j| \, d\mu = \sum_{j=1}^\infty \int_\Omega |f_j| \, d\mu < \infty$, it holds $\sum_{j=1}^\infty |f_j| < \infty$ μ-almost everywhere. □

(b) Let $\{r_k\}_{k\in\mathbb{N}}$ be an ordering of $\mathbb{Q}\cap[0,1]$ and $(a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be such that $\sum_{k=1}^\infty a_k$ is absolutely convergent. Show that $\sum_{k=1}^\infty a_k|x-r_k|^{-1/2}$ is absolutely convergent for almost every $x \in [0,1]$ (with respect to the Lebesgue measure).

Solution: We apply part (a) to the functions $f_k(x) = a_k|x-r_k|^{-1/2}$ with μ equal to the Lebesgue measure. It holds

$$\int_0^1 |f_k(x)| dx = |a_k| \int_{r_k}^1 \frac{1}{\sqrt{x-r_k}} dx + \int_0^{r_k} \frac{1}{\sqrt{r_k - x}} dx$$
$$= 2|a_k|(\sqrt{1-r_k} + \sqrt{r_k}) \leq 2\sqrt{2}|a_k| \ .$$

Therefore, $\sum_{k=1}^\infty \int_0^1 |f_k| dx \leq 2\sqrt{2} \sum_{k=1}^\infty |a_k| < \infty$ by assumption and with part (a) of the exercise, the result follows. □

Exercise 9.3.

Find an example of a continuous bounded function $f \colon [0,\infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$ and

$$\int_0^\infty |f(x)|^p dx = \infty \ ,$$

for all $p > 0$.

Solution: The function $f \colon [0,\infty) \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{\log(2+x)}$$

is continuous, bounded by $f(x) \leq \log(2)^{-1}$ and $\lim_{x\to\infty} f(x) = 0$. Since $\log(2+x) \leq p(2+x)^{\frac{1}{p}}$ for all $p > 0$, we get

$$\left|\frac{1}{\log(2+x)}\right|^p \geq \frac{1}{p^p(2+x)} \ ,$$

which is not integrable over $[0,\infty)$. □

Exercise 9.4.
Let $f, g : \Omega \to \overline{\mathbb{R}}$ be μ-summable functions and assume that

$$\int_A f d\mu \leq \int_A g d\mu$$

for all μ-measurable subsets $A \subset \Omega$. Show that $f \leq g$ μ-almost everywhere. Moreover, conclude that, if

$$\int_A f d\mu = \int_A g d\mu$$

for all μ-measurable subsets $A \subset \Omega$, then $f = g$ μ-almost everywhere.

Solution: Define $A := \{g < f\}$ and $A_n := \{g + \frac{1}{n} \leq f\}$ for all $n \in \mathbb{N}$. Notice that $\bigcup_{n\in\mathbb{N}} A_n = A$ and that A_n, A are measurable. Therefore, we find:

$$\frac{1}{n}\mu(A_n) + \int_{A_n} g d\mu = \int_{A_n} \left(g + \frac{1}{n}\right) d\mu \leq \int_{A_n} f d\mu \leq \int_{A_n} g d\mu.$$

Comparing the LHS and the RHS, we obtain $\mu(A_n) = 0$. Therefore, by continuity of the measure, we get $\mu(A) = 0$.

The second part of the exercise follows trivially from the first part. □

Exercise 9.5.
Let $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ be Lebesgue measurable functions. Find examples for the following statements.

(a) $f_n \to 0$ uniformly, but not $\int |f_n| dx \to 0$.

Solution: The functions $f_n = \frac{1}{n} \cdot \chi_{[0,n]}$ are easily an example. □

(b) $f_n \to 0$ pointwise and in measure, but neither $f_n \to 0$ uniformly nor $\int |f_n| dx \to 0$.

Solution: The functions $f_n = n \cdot \chi_{[\frac{1}{n}, \frac{2}{n}]}$ are an example. All properties are trivially true except for convergence in measure. For this, for all $\varepsilon > 0$, notice that

$$\mathcal{L}^1(|f_n - 0| > \varepsilon) \leq \frac{1}{n} \to 0.$$ □

(c) $f_n \to 0$ pointwise, but not in measure.

Solution: The functions $f_n = \chi_{[n,n+1]}$ are an example. They clearly not converge in measure as the limit would necessarily have to agree with the pointwise limit, since appropriate subsequences of a sequence converging in measure converge pointwise to the same limit. However it is obvious that this is not the case here. □

Exercise 9.6.
Let $f : [0,1] \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Show that for a set $E \subset [0,1]$ of positive measure it holds that

$$f(x) \leq \int_{[0,1]} f(y)\, d\mathcal{L}^1(y)$$

for every $x \in E$.

Solution: We argue by contradiction: if the statement is not true, then for almost every $x \in [0,1]$ it holds that $f(x) > J$, where $J := \int_{[0,1]} f(y)\, d\mathcal{L}^1(y)$. Let

$$A_n := \left\{ x \in [0,1] \,:\, f(x) > J + \frac{1}{n} \right\},$$

which is clearly a measurable set for each $n \in \mathbb{Z}^+$. It follows from our assumption that $\mathcal{L}^1([0,1] \setminus \bigcup_{n\geq 1} A_n) = 0$ and in particular $\mathcal{L}^1(A_n) > 0$ for some n. Then

$$\begin{aligned} J = \int_{[0,1]} f(y)\, d\mathcal{L}^1(y) &= \int_{A_n} f(y)\, d\mathcal{L}^1(y) + \int_{A_n^c} f(y)\, d\mathcal{L}^1(y) \\ &\geq \left(J + \frac{1}{n}\right) \mathcal{L}^1(A_n) + J\mathcal{L}^1(A_n^c) = J + \frac{1}{n}, \end{aligned}$$

which is clearly a contradiction. □