Exercise 10.1. ♣
Which of the following statements are true?

(a) Let $\{f_k\}$ be a sequence of nonnegative $\mathcal{L}^1$-measurable functions on $\mathbb{R}$ converging uniformly to a function f. Then $\lim_{k\to\infty}\int_{\mathbb{R}} f_k\, d\mathcal{L}^1$ exists and

$$\int_{\mathbb{R}} f\, d\mathcal{L}^1 \leq \lim_{k\to\infty}\int_{\mathbb{R}} f_k\, d\mathcal{L}^1.$$

Solution: ✗ Even with uniform convergence the limit may not exist: consider for example $f_k(x) = k^{-1}\chi_{[0,k]}(x)$ for k even and $f_k \equiv 0$ for k odd.

(b) Let $f_k : [0,1] \to [0,1]$ be $\mathcal{L}^1$-measurable functions for $k = 1, 2, \dots$ and suppose that $f_k \to f$ almost everywhere. Then $\lim_{k\to\infty}\int_{[0,1]} f_k\, d\mathcal{L}^1$ exists and

$$\int_{[0,1]} f\, d\mathcal{L}^1 \leq \lim_{k\to\infty}\int_{[0,1]} f_k\, d\mathcal{L}^1.$$

Solution: ✓ This is a consequence of the Dominated Convergence Theorem and actually equality always holds.

(c) Let f be $\mathcal{L}^1$-summable on $\mathbb{R}$ and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ be $\mathcal{L}^1$-measurable subsets of $\mathbb{R}$. Then the limit $\lim_{n\to\infty}\int_{E_n} f\, d\mathcal{L}^1$ exists.

Solution: ✓ It exists and is equal to $\int_{\bigcup_{n\geq 1} E_n} f\, d\mathcal{L}^1$ by the Dominated Convergence Theorem.

(d) Let $\{f_n\}$ be a sequence of continuous Lebesgue-summable functions on $[0,\infty)$ which converges to a Lebesgue-summable function f. Then

$$\lim_{n\to\infty}\int_{[0,\infty)} |f_n(x) - f(x)|\, \mathcal{L}^1(x) = 0.$$

Solution: ✗ Take for example

$$f_n(x) = \begin{cases} \frac{1}{n} - \frac{x}{n^2}, & x \in (0,n) \\ 0, & \text{otherwise} \end{cases}.$$

Exercise 10.2.
Let $f : \mathbb{R} \to [0,+\infty]$ be $\mathcal{L}^1$-measruable. Assume that for all $n \geq 1$,

$$\int_{\mathbb{R}} \frac{n^2}{n^2+x^2}|f(x)|\, d\mathcal{L}^1(x) \leq 1.$$

Show that

$$\int_{\mathbb{R}} |f| \, d\mathcal{L}^1 \leq 1.$$

Solution: Observe that we can write the integrand as

$$g_n(x) := \frac{n^2}{n^2 + x^2} |f(x)| = \left(1 - \frac{x^2}{n^2 + x^2}\right) |f(x)|,$$

which shows that for a fixed x it is monotonically increasing in n. Moreover the pointwise limit of $g_n(x)$ is clearly equal to $|f(x)|$ for every x. Therefore we may apply the Monotone Convergence Theorem and get

$$\int_{\mathbb{R}} |f(x)| \, d\mathcal{L}^1(x) = \lim_{n\to\infty} \int_{\mathbb{R}} g_n(x) \, d\mathcal{L}^1(x) = \lim_{n\to\infty} \int_{\mathbb{R}} \frac{n^2}{n^2 + x^2} |f(x)| \, d\mathcal{L}^1(x) \leq 1. \qquad \square$$

Exercise 10.3.
Compute the limit

$$\lim_{n\to\infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x} \, dx.$$

Solution: Notice that we can write the integrals as $\int_{[0,\infty)} f_n \, dx$, where

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n e^{-2x} \chi_{[0,n]}(x).$$

We claim that this sequence of functions is monotonically increasing: given n, it is clear that $f_n(x) \leq f_{n+1}(x)$ for $x > n$, so we may focus on the case $0 \leq x \leq n$ and forget about the characteristic function. We have to show that

$$\left(1 + \frac{x}{n}\right)^n \leq \left(1 + \frac{x}{n+1}\right)^{n+1}.$$

Taking both sides to the power $1/n$, this is equivalent to

$$1 + \frac{x}{n} \leq \left(1 + \frac{x}{n+1}\right)^{\frac{n+1}{n}}$$

Letting $y := x/(n+1)$ and $\alpha := \frac{n+1}{n} > 1$, we can rewrite this inequality as

$$1 + \alpha y \leq (1 + y)^{\alpha},$$

which is well-known (it can be easily shown for example by applying the Mean Value Theorem to the function $t \mapsto (1+t)^{\alpha}$ between 0 and y).

Let us now compute the pointwise limit of the increasing sequence f_n. Again we can ignore the characteristic function because for every x it becomes eventually 1.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n e^{-2x} = e^x \cdot e^{-2x} = e^{-x}.$$

We can now conclude by applying the Monotone Convergence Theorem:

$$\lim_{n\to\infty}\int_{[0,n]}\left(1+\frac{x}{n}\right)^n e^{-2x}\,dx = \lim_{n\to\infty}\int_{[0,\infty)} f_n(x)\,dx = \int_{[0,\infty)}\lim_{n\to\infty} f_n(x)\,dx = \int_{[0,\infty)} e^{-x}\,dx = 1. \quad \square$$

Exercise 10.4. ★
Let f_k, f be $\mathcal{L}^1$-summable functions on $\mathbb{R}$ which are nonnegative $\mathcal{L}^1$-almost everywhere and satisfy the following additional hypotheses:

- $\liminf_{k\to\infty} f_k(x) \geq f(x)$ for $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$.
- $\limsup_{k\to\infty}\int_{\mathbb{R}} f_k(x)\,dx \leq \int_{\mathbb{R}} f(x)\,dx$.

Show that

$$\lim_{k\to\infty}\int_{\mathbb{R}} |f_k(x) - f(x)|\,dx = 0.$$

Solution: Let f^+ and f^- denote the positive and negative part of a function f, respectively. Arguing as in the proof of the Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}}\liminf_{k\to\infty}\left(f^+ - (f-f_k)^+\right) \leq \liminf_{k\to\infty}\int_{\mathbb{R}} f^+ - (f-f_k)^+ \tag{1}$$

by applying Fatou's lemma to the functions $f^+ - (f-f_k)^+$ which are nonnegative, since $(f-f_k)^+ = \max\{0, f-f_k\} \leq \max\{0, f\} = f^+$. Since f^+ is summable (because f is), by linearity we can subtract $\int_{\mathbb{R}} f^+$ on both sides of (1) and obtain

$$\int_{\mathbb{R}}\limsup_{k\to\infty}(f-f_k)^+ \geq \limsup_{k\to\infty}\int_{\mathbb{R}}(f-f_k)^+. \tag{2}$$

However notice that $\liminf_{k\to\infty} f_k - f \geq 0$, which implies that $\limsup_{k\to\infty} f - f_k \leq 0$, and applying $(\cdot)^+$ on both sides and using the monotonicity of the lim sup—i.e. the fact that if φ is an increasing function then $\limsup_{k\to\infty}\varphi(a_k) = \varphi\left(\limsup_{k\to\infty} a_k\right)$—we get that $\limsup_{k\to\infty}(f-f_k)^+ \leq 0$. The opposite inequality is trivial, so we actually have $\limsup_{k\to\infty}(f-f_k)^+ = 0$. Inserting this into (2) we find

$$\limsup_{k\to\infty}\int_{\mathbb{R}}(f-f_k)^+ = 0. \tag{3}$$

On the other hand, observe that $(f-f_k)^+ - (f-f_k)^- = f - f_k$. Hence

$$\limsup_{k\to\infty}\int_{\mathbb{R}}(f-f_k)^- = \limsup_{k\to\infty}\int_{\mathbb{R}}(f-f_k)^+ - f + f_k \leq \limsup_{k\to\infty}\int_{\mathbb{R}}(f-f_k)^+ + \limsup_{k\to\infty}\int_{\mathbb{R}} f_k - f = 0 \tag{4}$$

by (3) and the second condition. Adding (3) and (4) and recalling that $|f - f_k| = (f - f_k)^+ + (f - f_k)^-$, we finally get

$$\limsup_{k\to\infty} \int_{\mathbb{R}} |f - f_k| \leq \limsup_{k\to\infty} \int_{\mathbb{R}} (f - f_k)^+ + \limsup_{k\to\infty} \int_{\mathbb{R}} (f - f_k)^- = 0.$$

□

Exercise 10.5. ★

Let $0 < m < M < \infty$ be two real numbers and let $f : [0,1] \to \mathbb{R}$ be a measurable function satisfying $m \leq f(x) \leq M$ for almost every $x \in [0,1]$. Show that

$$\left(\int_{[0,1]} f(x)\,dx\right)\left(\int_{[0,1]} \frac{1}{f(x)}\,dx\right) \leq \frac{(m+M)^2}{4mM}$$

and characterize all functions for which equality holds.

Solution: Since $f(x)$ satisfies the inequality $m \leq f(x) \leq M$ pointwise almost everywhere, it holds that

$$(f(x) - m)(M - f(x)) \geq 0.$$

Expanding and dividing by $f(x) > 0$ we obtain

$$\frac{mM}{f(x)} + f(x) \leq m + M. \tag{5}$$

Integrating we obtain the inequality

$$mM \int_{[0,1]} \frac{1}{f(x)}\,dx + \int_{[0,1]} f(x)\,dx \leq m + M. \tag{6}$$

By applying the arithmetic-geometric inequality on the left we get

$$2\left(mM \int_{[0,1]} \frac{1}{f(x)}\,dx \int_{[0,1]} f(x)\,dx\right)^{1/2} \leq mM \int_{[0,1]} \frac{1}{f(x)}\,dx + \int_{[0,1]} f(x)\,dx \leq m + M. \tag{7}$$

Finally, squaring and rearranging we prove the desired inequality:

$$\int_{[0,1]} \frac{1}{f(x)}\,dx \cdot \int_{[0,1]} f(x)\,dx \leq \frac{(m+M)^2}{4mM}.$$

In order to have equality, we must have equality in the arithmetic-geometric inequality (7),

$$mM \int_{[0,1]} \frac{1}{f(x)}\,dx = \int_{[0,1]} f(x)\,dx \tag{8}$$

and in (6). This implies that equality (5) holds almost everywhere, which means that $f(x) \in \{m, M\}$ for almost every $x \in [0,1]$. Letting A and B be the sets where $f = m$ and $f = M$, respectively, by inserting into (8) we find

$$mM\left(\frac{\lambda}{m} + \frac{1-\lambda}{M}\right) = \lambda m + (1-\lambda)M,$$

where $\lambda = \mathcal{L}^1(A) = 1 - \mathcal{L}^1(B)$. A simple computation then shows that $\lambda = 1/2$, so equality holds when $f \equiv m$ almost everywhere in a measurable set of half the measure and $f \equiv M$ almost everywhere in its complement. □

Exercise 10.6.
For all $n \in \mathbb{N}$, let $f_n\colon [0,1] \to \mathbb{R}$ be defined by:

$$f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2}.$$

Prove that:

(a) $f_n(x) \leq \frac{1}{\sqrt{x}}$ on $(0,1]$ for all $n \geq 1$.

Solution: We would like to show that $\frac{n\sqrt{x}}{1+n^2x^2} \leq \frac{1}{\sqrt{x}}$. This is equivalent to

$$nx \leq 1 + n^2x^2 \Leftrightarrow (1-nx)^2 + nx \geq 0$$

which is true for all $x \in [0,1]$. □

(b) $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

Solution: Let us start with the following observation

$$\frac{n\sqrt{x}}{1+n^2x^2} \leq \frac{n\sqrt{x}}{n^2x^2} \leq \frac{1}{nx\sqrt{x}}.$$

Therefore, it holds

$$\lim_{n\to\infty} f_n(x) = 0$$

pointwise on $(0,1]$.

By (a), we know that the sequence f_n is always smaller than $g = \frac{1}{\sqrt{x}}$. Since g is Lebesgue integrable on $[0,1]$, we deduce by Lebegue's dominated convergence theorem and the pointwise convergence to 0 that:

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$ □