Exercise 11.1. ♣

(a) Let $\{f_k\}$ be a sequence of $\mathcal{L}^1$-measurable functions on $[0,1]$ converging a.e. to a function f and such that $|f_k| \leq 100$ a.e. for each k. Is it true that

$$\lim_{k\to\infty} \int_{[0,1]} |f_k - f|\, dx = 0?$$

Solution: Yes: we can apply Dominated Convergence with the dominating function $g \equiv 100$.

(b) Compute the limit

$$\lim_{k\to\infty} k \int_0^\infty e^{-kx} \sqrt{|\cos(x)|}\, dx.$$

Solution: Make the change of variables $y = kx$. Then the integral becomes

$$\int_0^\infty k e^{-kx} \sqrt{|\cos(x)|}\, dx = \int_0^\infty e^{-y} \sqrt{\left|\cos\left(\frac{y}{k}\right)\right|}\, dy.$$

The new integrands are all bounded below by $g(y) := e^{-y}$, which is summable, and therefore we may pass to the limit using Dominated Convergence. The pointwise limit is

$$\lim_{k\to\infty} e^{-y} \sqrt{\left|\cos\left(\frac{y}{k}\right)\right|} = e^{-y}\sqrt{|\cos(0)|} = e^{-y},$$

thus

$$\lim_{k\to\infty} \int_0^\infty k e^{-kx} \sqrt{|\cos(x)|}\, dx = \lim_{k\to\infty} \int_0^\infty e^{-y} \sqrt{\left|\cos\left(\frac{y}{k}\right)\right|}\, dy = \int_0^\infty e^{-y}\, dy = 1.$$

(c) What is the value of the limit

$$\lim_{k\to\infty} \int_0^\infty e^{-x^k}\, dx?$$

(A) 0. (B) 1. (C) ∞. (D) None of the previous answers is correct.

Solution: The correct answer is (B). The sequence e^{-x^k} is monotonically increasing to 1 for $x \in (0,1)$ and monotonically decreasing to 0 for $x > 1$. Therefore the function g defined as

$$g(x) := \begin{cases} 1, & \text{if } x \leq 1 \\ e^{-x}, & \text{if } x > 1 \end{cases}$$

dominates the sequence $\{e^{-x^k}\}$ and is summable. Passing to the limit,

$$\lim_{k\to\infty} \int_0^\infty e^{-x^k}\, dx = \int_0^\infty \lim_{k\to\infty} e^{-x^k}\, dx = \int_0^\infty \chi_{[0,1]}(x)\, dx = 1.$$

(d) Let $f : \Omega \to [0,1]$ be a μ-measurable function with $\int_\Omega f \, d\mu > 0$. Is it true that

$$\lim_{k\to\infty} \int_\Omega f^{1/k} \, d\mu > 0?$$

Solution: Yes: first notice that the set $E := \{x \in \Omega \,:\, f(x) > 0\}$ has $\mu(E) > 0$, since otherwise the integral of f would be zero. For each $x \in E$, $\lim_{k\to\infty} f(x)^{1/k} = 1$ and moreover this sequence is monotonically increasing. Thus we may pass to the limit using the Monotone Convergence Theorem:

$$\lim_{k\to\infty} \int_\Omega f^{1/k} \, d\mu = \int_\Omega \lim_{k\to\infty} f^{1/k} \, d\mu = \int_E \lim_{k\to\infty} f^{1/k} \, d\mu = \int_E 1 \, d\mu = \mu(E) > 0.$$

Exercise 11.2.
Compute the limit

$$\lim_{n\to\infty} \int_a^{+\infty} \frac{n}{1+n^2x^2} \, dx$$

for every $a \in \mathbb{R}$.

Hint: recall that $\arctan x$ is a primitive of $\frac{1}{1+x^2}$.

Solution: Observe that $\frac{n}{1+n^2x^2} \leq \frac{1}{nx^2} \leq \frac{1}{x^2}$ for $x > 0$. If $a > 0$, then since the function $\frac{1}{x^2}$ is integrable on $(a, +\infty)$, we may apply Lebesgue's dominated convergence theorem and deduce that

$$\lim_{n\to\infty} \int_a^{+\infty} \frac{n}{1+n^2x^2} \, dx = \int_a^{+\infty} \lim_{n\to\infty} \frac{n}{1+n^2x^2} \, dx = \int_a^{+\infty} 0 \, dx = 0.$$

For $a = 0$ we can use the change of variables $y = nx$ and see that the integral is actually independent of n:

$$\int_0^{+\infty} \frac{n}{1+n^2x^2} \, dx = \int_0^{+\infty} \frac{1}{1+y^2} \, dy = \arctan y \Big|_0^{+\infty} = \arctan(+\infty) - \arctan(0) = \frac{\pi}{2}.$$

Finally for $a < 0$ we get, by using the fact that the integrand is even:

$$\begin{aligned} \int_a^{+\infty} \frac{n}{1+n^2x^2} \, dx &= \int_{-\infty}^{+\infty} \frac{n}{1+n^2x^2} \, dx - \int_{-\infty}^{a} \frac{n}{1+n^2x^2} \, dx \\ &= 2\int_0^{+\infty} \frac{n}{1+n^2x^2} \, dx - \int_{-a}^{+\infty} \frac{n}{1+n^2x^2} \, dx. \end{aligned}$$

Thus using the two previous cases we deduce that the limit is π. □

Exercise 11.3.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ be μ-measurable and $f : \Omega \to [0, +\infty]$ be μ-summable. For all μ-measurable subsets $A \subset \Omega$ define (see Section 3.5 in the Lecture Notes)

$$\nu(A) = \int_A f d\mu.$$

(a) Prove that ν is a pre-measure on the σ-algebra of μ-measurable sets, hence we can define its Carathéodory-Hahn extension $\nu : \mathcal{P}(\Omega) \to [0, +\infty]$.

Solution: Obviously we have that $\nu(\emptyset) = 0$. Now consider a family $\{A_k\}_{k\in\mathbb{N}}$ of pairwise disjoint μ-measurable sets with $A = \bigcup_{k\in\mathbb{N}} A_k$. For all $k \in \mathbb{N}$, consider the function $f_k : \Omega \to [0, +\infty]$ defined as $f_k = f(\chi_{A_0} + \chi_{A_1} + \ldots + \chi_{A_k})$. Note that $f_k \leq f_{k+1}$ for all $k \in \mathbb{N}$ and $f_k \xrightarrow{k\to\infty} f\chi_A$ pointwise. Hence by Beppo Levi's Theorem we get

$$\begin{aligned}\nu(A) &= \int_A f d\mu = \int_\Omega f\chi_A d\mu = \int_\Omega \lim_{k\to\infty} f_k \, d\mu = \lim_{k\to\infty} \int_\Omega f_k d\mu \\ &= \lim_{k\to\infty} \sum_{i=0}^{k} \int_\Omega f\chi_{A_i} d\mu = \lim_{k\to\infty} \sum_{i=0}^{k} \int_{A_i} f d\mu = \sum_{k\in\mathbb{N}} \nu(A_k),\end{aligned}$$

where we used Theorem 3.1.15 and Lemma 3.1.17 of the Lecture Notes. Hence we proved that ν is a pre-measure and therefore can be extended to a measure $\nu : \mathcal{P}(\Omega) \to [0, +\infty]$. Moreover the σ-algebra Σ_ν of ν-measurable sets contains the σ-algebra Σ_μ of μ-measurable sets. □

(b) Show that ν is a Radon measure.

Solution: First note that ν is a Borel measure since μ is a Borel measure and $\Sigma_\nu \supset \Sigma_\mu$.

Now let us prove that ν is Borel regular. First consider any μ-measurable subset $A \subseteq \Omega$. Since μ is Borel regular, there exists a Borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$. We can also suppose that $\mu(B \setminus A) = 0$, for example by obtaining first $B_i \supseteq A \cap Q_i$, where $\{Q_i\}$ is the standard partition of $\mathbb{R}^n$ into unit cubes, and then setting $B = \bigcup_i B_i$. Hence it holds

$$\nu(A) = \int_A f d\mu = \int_B f d\mu - \int_{B\setminus A} f d\mu = \int_B f d\mu = \nu(B),$$

where we used that $\mu(B \setminus A) = 0$ and Corollary 3.1.18. Now let $A \subset \Omega$ be any set. By definition of Carathéodory-Hahn extension, there exist μ-measurable sets $A_k \supset A$ such that $\nu(A) = \lim_{k\to\infty} \nu(A_k)$. For what we proved just above, there exist Borel sets $B_k \supset A_k \supset A$ such that $\nu(B_k) = \nu(A_k)$. Then define the Borel set $B = \bigcap_{k\in\mathbb{N}} B_k$, for which it easily holds $\nu(B) = \nu(A)$. This proves that ν is Borel regular.

Let $K \subset \Omega$ be any compact set, then

$$\nu(K) = \int_K f d\mu < +\infty,$$

where we used that f is μ-summable. This concludes the proof that ν is a Radon measure. □

(c) Prove that $\Sigma_\nu \supseteq \Sigma_\mu$ and that ν is absolutely continuous with respect to μ, that is, if $\mu(A) = 0$ then $\nu(A) = 0$.

Solution: We have already observed that $\Sigma_\nu \supset \Sigma_\mu$. Moreover, if $\mu(A) = 0$ for a subset $A \subset \Omega$, then $\nu(A) = 0$ by Corollary 3.1.18. This proves that ν is absolutely continuous with respect to μ. □

Exercise 11.4.
Prove the following assertions.

(a) Let $f \colon [a, +\infty) \to \mathbb{R}$ be a locally bounded function and locally Riemann integrable. Then f is $\mathcal{L}^1$-summable if and only if f is absolutely Riemann integrable in the generalized sense (namely $\mathcal{R}\int_a^\infty |f(x)|dx = \lim_{j\to\infty} \mathcal{R}\int_a^j |f(x)|dx$ exists and it is finite) and in this case

$$\int_{[a,+\infty)} f(x)d\mathcal{L}^1 = \mathcal{R}\int_a^\infty f(x)dx = \lim_{j\to+\infty} \mathcal{R}\int_a^j f(x)dx.$$

Solution: See proof of Exercise 3.6.7 (2) in the Lecture Notes.

(b) Let $f \colon [0, +\infty) \to \mathbb{R}$ be the function $f(x) = \frac{\sin x}{x}$, which is locally bounded and locally Riemann integrable. Show that f is Riemann integrable, i.e. $\mathcal{R}\int_0^\infty f(x)dx < +\infty$ but not absolutely Riemann integrable, i.e. $\mathcal{R}\int_0^\infty |f(x)|dx = \infty$. Hence f is not $\mathcal{L}^1$-summable.

Solution: In what follows we write $\int$ for the Riemann integral $\mathcal{R}\int$. We have that

$$\int_0^j \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^j \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \Big[-\frac{\cos x}{x}\Big]_1^j - \int_1^j \frac{\cos x}{x^2}dx.$$

Now note that $\int_0^1 \frac{\sin x}{x}dx < +\infty$, $\big[-\frac{\cos x}{x}\big]_1^j = \cos 1 - \cos j/j$ and

$$\left|\int_1^j \frac{\cos x}{x^2}dx\right| \le \int_1^j \frac{|\cos x|}{x^2}dx \le \int_1^j \frac{1}{x^2}dx = 1 - \frac{1}{j}.$$

Hence $\lim_{j\to\infty} \int_0^j \frac{\sin x}{x}dx$ exists and is finite. On the other hand we have that

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \ge \sum_{k\in\mathbb{N}} \int_{\pi k}^{\pi(k+1)} \left|\frac{\sin x}{x}\right| dx \ge \sum_{k\in\mathbb{N}} \frac{1}{k+1}\cdot\frac{1}{2}\cdot\frac{\pi}{3} = +\infty.$$ □

Exercise 11.5.
This exercise is a more general version of Theorem 3.4.1 from the lecture notes.

(a) Let μ be a Radon measure on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a μ-measurable subset. Consider a function $f \colon \Omega \times (a, b) \to \mathbb{R}$, for some interval $(a, b) \subset \mathbb{R}$, such that:

- the map $x \mapsto f(x, y)$ is μ-summable for all $y \in (a, b)$;
- the map $y \mapsto f(x, y)$ is differentiable in (a, b) for every $x \in \Omega$;

- there is a μ-summable function $g\colon \Omega \to [0,\infty]$ such that $\sup_{a<y<b}|\frac{\partial f}{\partial y}(x,y)| \leq g(x)$ for all $x \in \Omega$.

Then $y \mapsto \int_\Omega f(x,y)d\mu(x)$ is differentiable in (a,b) with

$$\frac{d}{dy}\left(\int_\Omega f(x,y)d\mu(x)\right) = \int_\Omega \frac{\partial f}{\partial y}(x,y)d\mu(x)$$

for all $y \in (a,b)$.

Solution: Fix $y \in (a,b)$, let $\{h_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers converging to 0 and consider the μ-summable function

$$g_k(x) = \frac{f(x,y+h_k) - f(x,y)}{h_k}$$

for all k large enough so that $y + h_k \in (a,b)$. Note that $g_k(x) \to \frac{\partial f}{\partial y}(x,y)$ pointwise as $k \to \infty$. Moreover, by the mean value theorem, we have that

$$|g_k(x)| \leq \sup_{a<y'<b}\left|\frac{\partial f}{\partial y}(x,y')\right| \leq g(x).$$

We also have that $\frac{\partial f}{\partial y}(\cdot,y)$ is μ-measurable, since it is the pointwise limit of μ-measurable functions. Thus we can apply the Dominated Convergence Theorem, obtaining that $\frac{\partial f}{\partial y}(\cdot,y)$ is μ-summable and

$$\begin{aligned}\int_\Omega \frac{\partial f}{\partial y}(x,y)d\mu(x) &= \lim_{k\to\infty}\int_\Omega g_k(x)d\mu(x) = \lim_{k\to\infty}\frac{\int_\Omega f(x,y+h_k)d\mu(x) - \int_\Omega f(x,y)d\mu(x)}{h_k} \\ &= \frac{d}{dy}\int_\Omega f(x,y)d\mu(x),\end{aligned}$$

which concludes the proof. □

(b) ★ Compute the integral

$$\phi(y) := \int_{(0,\infty)} e^{-x^2-y^2/x^2}d\mathcal{L}^1(x)$$

for all $y > 0$.

Hint: use part (a) to obtain that ϕ solves the Cauchy problem

$$\begin{cases}\phi'(y) = -2\phi(y) & \text{for } y > 0 \\ \lim_{y\to 0^+}\phi(y) = \sqrt{\pi}/2. & \end{cases}$$

Solution: First note that $e^{-x^2-y^2/x^2} \leq e^{-x^2}$ is $\mathcal{L}^1$-summable for all $y > 0$ and $y \mapsto e^{-x^2-y^2/x^2}$ is differentiable in $(0,+\infty)$ for all $x > 0$. Moreover, for all $x, y > 0$, we have that

$$\left|\frac{\partial}{\partial y}e^{-x^2-y^2/x^2}\right| = \frac{2y}{x^2}e^{-x^2-y^2/x^2} \leq \frac{2e^{-x^2}}{y}\cdot\frac{y^2}{x^2}e^{-y^2/x^2} \leq \frac{2e^{-x^2}}{y}e^{-1}.$$

Hence $\frac{\partial}{\partial y}e^{-x^2-y^2/x^2}$ is controlled by the $\mathcal{L}^1$-summable function e^{-x^2}/r for all $y > r$. Therefore we can apply part (a) and obtain that

$$\begin{aligned}\phi'(y) &= -\int_{(0,\infty)} \frac{2y}{x^2} e^{-x^2-y^2/x^2} d\mathcal{L}^1(x) \overset{t=y/x}{=} -2\int_{(0,\infty)} y\frac{t^2}{y^2} e^{-t^2-y^2/t^2} \frac{y}{t^2} d\mathcal{L}^1(t) \\ &= -2\int_{(0,\infty)} e^{-t^2-y^2/t^2} d\mathcal{L}^1(t) = -2\phi(y).\end{aligned}$$

Since $\int_0^\infty e^{-x^2} d\mathcal{L}^1(x) = \sqrt{\pi}/2$, ϕ satisfies the Cauchy problem

$$\begin{cases}\phi'(y) = -2\phi(y) & \text{for } y > 0 \\ \lim_{y\to 0^+}\phi(y) = \sqrt{\pi}/2, & \end{cases}$$

which has solution $\phi(y) = \sqrt{\pi}e^{-y}/2$. □

Exercise 11.6.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$ and $f, f_k : \Omega \to \overline{\mathbb{R}}$ μ-summable functions.

(a) Show that Vitali's Theorem implies Dominated Convergence Theorem.

Solution: Let $g : \Omega \to [0,\infty]$ be μ-summable and consider $|f_k| \leq g$ and $f_k \to f$ μ-almost everywhere, where $f, f_k : \Omega \to \overline{\mathbb{R}}$ are μ-measurable functions, for $k \in \mathbb{N}$.

Since $\mu(\Omega) < +\infty$, we have the convergence $f_k \overset{\mu}{\to} f$ (see Theorem 2.4.2 in the Lecture Notes). In addition, the f_k's are uniformly μ-summable. This is due to the monotonicity of the integral for $|f_k| \leq g$ and the absolute continuity of the integral of g (see theorem below). As a result, the conditions of Vitali's theorem are satisfied and it follows that $\lim_{k\to\infty}\int_\Omega |f_k - f| d\mu = 0$. □

Let us conclude by proving the absolute continuity of the integral of g (since in the lecture we used the Dominated Convergence Theorem), namely:

Theorem. *Let $g : \Omega \to \overline{\mathbb{R}}$ be μ-summable. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all μ-measurable subsets $A \subset \Omega$ with $\mu(A) < \delta$, it holds $\int_A |g| d\mu < \varepsilon$.*

Proof. Without loss of generality, assume that $g \geq 0$ and define $g_n := \min\{g, n\}$. Then g_n converges pointwise to g μ-a.e. and by monotone convergence we have $\lim_{n\to\infty}\int_\Omega g_n d\mu = \int_\Omega g d\mu$, in particular $\lim_{n\to\infty}\int_\Omega |g - g_n| d\mu = 0$.

Now let $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ such that $\int_\Omega |g - g_N| d\mu < \varepsilon/2$. Hence, choosing $\delta = \varepsilon/(2N)$, we deduce for all measurable subsets $A \subset \Omega$ with $\mu(A) < \delta$ that

$$\int_A |g| d\mu \leq \int_A |g - g_N| d\mu + \int_A |g_N| d\mu < \int_\Omega |g - g_N| d\mu + \mu(A)N < \varepsilon.$$

This indeed proves the absolute continuity of the integral. □

(b) Let $\Omega = [0,1]$ and $\mu = \mathcal{L}^1$. Give an example in which Vitali's Theorem can be applied but Dominated Convergence Theorem cannot, i.e., a dominating function does not exist.

Hint: look at the functions $f_n^k(x) = \frac{1}{x}\chi_{[\frac{n+k-1}{n2^{n+1}}, \frac{n+k}{n2^{n+1}})}(x)$ for $n \in \mathbb{N}$, $1 \leq k \leq n$.

Solution: For $n \in \mathbb{N}$, $1 \leq k \leq n$, consider the function $f_n^k(x) = \frac{1}{x}\chi_{[\frac{n+k-1}{n2^{n+1}}, \frac{n+k}{n2^{n+1}})}(x)$.

The sequence $\{f_n^k\}$ is uniformly μ-summable. Indeed, given $\varepsilon > 0$, choose $M \in \mathbb{N}$ with $1/M \leq \varepsilon$ and $\delta := 2^{-(M+1)}/M$, then for all $A \subset [0,1]$ with $\mu(A) < \delta$ we have:

- if $n \geq M$ and $1 \leq k \leq n$, then

$$\int_A |f_n^k(x)| dx \leq \int_0^1 |f_n^k(x)| dx \leq 2^{n+1} \cdot \frac{1}{n2^{n+1}} = \frac{1}{n} \leq \frac{1}{M} \leq \varepsilon;$$

- if $n < M$ and $1 \leq k \leq n$, then

$$\int_A |f_n^k(x)| dx \leq 2^{n+1}\delta = \frac{2^{n+1}}{M2^{M+1}} < \frac{1}{M} \leq \varepsilon.$$

Furthermore, $f_n^k \to 0$ converges pointwise and, as a result, converges in measure. Hence (f_n^k) satisfies the conditions of Vitali's theorem. However, a dominating function would have to be larger than $1/x$, which implies non-summability over $[0,1]$. □