Exercise 12.1. ♣

(a) The value of the limit

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-\pi x}\,dx$$

is

(A) 0. (B) $\frac{1}{\pi-1}$. ✓ (C) $\frac{2}{\pi-1}$. (D) 1.

(b) Is the following equality true?

$$\lim_{n\to\infty}\int_0^1 e^{\frac{x^2}{n}}\,dx = \int_0^1 \lim_{n\to\infty} e^{\frac{x^2}{n}}\,dx.$$

Solution: Yes, because the functions converge even uniformly.

(c) The value of the limit

$$\lim_{n\to\infty}\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx$$

is

(A) 0. ✓ (B) 1. (C) $+\infty$. (D) 2.

(d) Consider the following statements:

(i) If $f \in L^p([0,1])$ for all $p \in (1,\infty)$, then $f \in L^\infty([0,1])$.

(ii) If $1 \leq p < q < +\infty$, then $L^q([1,\infty)) \subseteq L^p([1,\infty))$.

Which of them are true?

(A) Both (i) and (ii).

(B) (i) but not (ii).

(C) (ii) but not (i).

(D) Neither (i) nor (ii). ✓

Solution: Both statements are false. For the first one, consider $f(x) = \ln x$, and for the second one, the function $g(x) = x^{-r}$ belongs to $L^q([1,\infty))$ but not to $L^p([1,\infty))$ if $\frac{1}{q} < r < \frac{1}{p}$.

Exercise 12.2.

Evaluate

$$\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}} \left(1-\sqrt{\sin x}\right)^n \cos x\,dx.$$

Solution: Since the integrands are nonnegative, by the Monotone Convergence applied to series, we can exchange the sum and the integral:

$$\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}}\left(1-\sqrt{\sin x}\right)^n\cos x\,dx=\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\left(1-\sqrt{\sin x}\right)^n\cos x\,dx.$$

For $x\neq 0$ (which we can ignore), n only appears in a geometric series of ratio $1-\sqrt{\sin x}\in(0,1)$, so the pointwise sum equals

$$\sum_{n=0}^{\infty}\left(1-\sqrt{\sin x}\right)^n\cos x=\frac{1}{1-\left(1-\sqrt{\sin x}\right)}\cos x=\frac{\cos x}{\sqrt{\sin x}}.$$

This function has as a primitive $2\sqrt{\sin x}$. Therefore the solution is

$$\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}}\left(1-\sqrt{\sin x}\right)^n\cos x\,dx=\left[2\sqrt{\sin x}\right]_0^{\frac{\pi}{2}}=2.$$

Exercise 12.3.
Let $1\leq p<\infty$. Show that if $\varphi\in L^p(\mathbb{R}^n)$ and φ is uniformly continuous, then

$$\lim_{|x|\to\infty}\varphi(x)=0.$$

Solution: Suppose, by contradiction, that there is $\varepsilon>0$ and a sequence $\{x_k\}$ with $|x_k|\to\infty$ and $|\varphi(x_k)|\geq\varepsilon$. Then by uniform continuity, there is $\delta>0$ such that for every $x\in B_\delta(x_k)$ we have $|\varphi(x)-\varphi(x_k)|\leq\varepsilon/2$, which implies that $|\varphi(x)|\geq\varepsilon/2$. Since $|x_k|\to\infty$, we can pass to a subsequence $\{x_{k_j}\}$ with $|x_{k_j}|>|x_{k_{j-1}}|+2\delta$. This implies in particular that for any $j\neq j'$, $|x_{k_j}-x_{k_{j'}}|>2\delta$, so that the balls $B_\delta(x_{k_j})$ and $B_\delta(x_{k_{j'}})$ are disjoint. Thus we get the following lower bound which shows that $\varphi\notin L^p(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n}|\varphi(x)|^p\,dx\geq\sum_{j=1}^{\infty}\int_{B_\delta(x_{k_j})}|\varphi(x)|^p\,dx\geq\sum_{j=1}^{\infty}\int_{B_\delta(x_{k_j})}\left(\frac{\varepsilon}{2}\right)^p\,dx=+\infty$$ □

Exercise 12.4.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega\subset\mathbb{R}^n$ a μ-measurable set.

(a) (Generalized Hölder inequality) Consider $1 \leq p_1, \ldots, p_k \leq \infty$ such that $\frac{1}{r} = \sum_{i=1}^k \frac{1}{p_i} \leq 1$. Show that, given functions $f_i \in L^{p_i}(\Omega, \mu)$ for $i = 1, \ldots, k$, it holds $\prod_{i=1}^k f_i \in L^r(\Omega, \mu)$ and

$$\left\| \prod_{i=1}^k f_i \right\|_{L^r} \leq \prod_{i=1}^k \|f_i\|_{L^{p_i}}.$$

Solution: We can suppose that all p_i are finite, since it is easy to deal with $p_i = \infty$ directly. We will prove the statement by induction. For $k = 1$ there is nothing to prove. For the induction step $k-1 \to k$, we know that $\frac{1}{r} - \frac{1}{p_k} = \frac{p_k - r}{p_k r} = \sum_{j=1}^{k-1} \frac{1}{p_j}$. By the induction hypothesis, we have that $\prod_{j=1}^{k-1} f_j \in L^{\frac{p_k r}{p_k - r}}(\Omega, \mu)$ together with the estimate

$$\left\| \prod_{j=1}^{k-1} f_j \right\|_{L^{\frac{p_k r}{p_k - r}}} \leq \prod_{j=1}^{k-1} \|f_j\|_{L^{p_j}}.$$

Now we apply Hölder's inequality to the functions $g_1 = \prod_{j=1}^{k-1} |f_j|^r$ and $g_2 = |f_k|^r$, with exponents $\frac{p_k}{p_k - r}$ and $\frac{p_k}{r}$ respectively:

$$\begin{aligned}
\int_\Omega \left(\prod_{j=1}^k |f_k| \right)^r &\leq \left(\int_\Omega \prod_{j=1}^{k-1} |f_j|^{r \frac{p_k}{p_k - r}} \right)^{\frac{p_k - r}{p_k}} \left(\int_\Omega |f_k|^{r \frac{p_k}{r}} \right)^{\frac{r}{p_k}} \\
&= \left\| \prod_{j=1}^{k-1} f_j \right\|^r_{L^{\frac{p_k r}{p_k - r}}} \|f_k\|^r_{L^{p_k}} \leq \prod_{j=1}^{k-1} \|f_j\|^r_{L^{p_j}} \cdot \|f_k\|^r_{L^{p_k}}.
\end{aligned}$$

This yields $\| \prod_{i=1}^k f_i \|_{L^r} \leq \prod_{i=1}^k \|f_i\|_{L^{p_i}}$, as we wanted to show. □

(b) Prove that, if $\mu(\Omega) < +\infty$, then $L^s(\Omega, \mu) \subseteq L^r(\Omega, \mu)$ for all $1 \leq r < s \leq +\infty$.

Solution: Fix $1 \leq r < s \leq +\infty$ and define $p = rs/(s-r)$, for which it holds $\frac{1}{s} + \frac{1}{p} = \frac{1}{r}$. If $\mu(\Omega) < +\infty$, then $g = 1 \in L^p(\Omega, \mu)$, hence we can apply part (a) and obtain that, for all $f \in L^r(\Omega, \mu)$, $f = f \cdot 1 \in L^r(\Omega, \mu)$, which proves the desired inclusion. □

(c) Show that the inclusion in part (b) is strict for all $1 \leq r < s \leq +\infty$.

Solution: For all $1 \leq r < +\infty$, consider the function $f\colon (0, 1/2) \to \mathbb{R}$ given by

$$f(x) = \left(\log^2\left(\frac{1}{x}\right) x^{1/r} \right)^{-1}.$$

Note that $f \in L^r$ since

$$\begin{aligned}
\int_0^{1/2} \left(\log^2\left(\frac{1}{x}\right) x^{1/r} \right)^{-r} dx &= \lim_{\varepsilon \to 0} \int_\varepsilon^{1/2} \left(\log^{2r}\left(\frac{1}{x}\right) x \right)^{-1} \\
&= \lim_{\varepsilon \to 0} \left[\frac{1}{(2r-1)\log^{2r-1}(1/x)} \right]_\varepsilon^{1/2} = \frac{1}{(2r-1)\log^{2r-1}(2)}.
\end{aligned}$$

On the other hand $f \notin L^s$ for all $s > r$: in this case we can choose $0 < t < \frac{1}{r} - \frac{1}{s}$ and estimate $\log^2\left(\frac{1}{x}\right) \leq Cx^{-t}$ with a constant $C > 0$. Then follows

$$\left(\log^2\left(\frac{1}{x}\right) x^{1/r}\right)^{-1} \geq \frac{1}{C} x^{t-\frac{1}{r}}$$

with $s\left(t - \frac{1}{r}\right) < -1$, which is not integrable. □

Exercise 12.5. ★
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$. Consider a function $f\colon \Omega \to \overline{\mathbb{R}}$ such that $fg \in L^1(\Omega,\mu)$ for all $g \in L^p(\Omega,\mu)$. Prove that $f \in L^q(\Omega,\mu)$ for all $q \in [1, p')$, where $p' = \frac{p}{p-1}$ is the conjugate of p.

Solution: First note that, taking $g = 1 \in L^p(\Omega,\mu)$, we get that $f \in L^1(\Omega,\mu)$. Hence we can consider the function $g = |f|^{1/p} \in L^p(\Omega,\mu)$ and we get that $|f|^{1+1/p} \in L^1(\Omega,\mu)$. Therefore we can choose $g = |f|^{1/p+1/p^2} \in L^p(\Omega,\mu)$ and get that $|f|^{1+1/p+1/p^2} \in L^1(\Omega,\mu)$.

Repeating again the same argument by induction, we get that $|f|^{p_n} \in L^1(\Omega,\mu)$ for all $n \in \mathbb{N}$, where $p_n = 1 + \frac{1}{p} + \cdots + \frac{1}{p^n} = \frac{1-1/p^{n+1}}{1-1/p}$. In particular we have that $f \in L^{p_n}(\Omega,\mu)$ for all $n \in \mathbb{N}$, which implies that $f \in L^q(\Omega,\mu)$ for all $1 \leq q \leq p_n$ by Exercise 12.4 (b). Now note that $p_n \to p'$ as $n \to \infty$, thus $f \in L^q(\Omega,\mu)$ for all $1 \leq q < p'$, as desired. □

Exercise 12.6.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set.

(a) Show that any $f \in \bigcap_{p\in\mathbb{N}^*} L^p(\Omega,\mu)$ with $\sup_{p\in\mathbb{N}^*} \|f\|_{L^p} < +\infty$ lies in $L^\infty(\Omega,\mu)$.
Hint: Tchebychev's inequality.

Solution: Let $C = \sup_{p\in\mathbb{N}^*}\|f\|_{L^p}$ and $\varepsilon > 0$. Using Tchebychev' inequality, we have:

$$\mu(\{|f| \geq C+\varepsilon\}) = \mu(\{|f|^p \geq (C+\varepsilon)^p\}) \leq \frac{1}{(C+\varepsilon)^p}\int_\Omega |f|^p d\mu$$
$$\leq \left(\frac{C}{C+\varepsilon}\right)^p \to 0\ , \quad \text{as } p \to \infty.$$

Hence $\mu(\{|f| \geq C+\varepsilon) = 0$ and we deduce $f \in L^\infty$. Since $\varepsilon > 0$ was arbitrary, by

$$\mu(\{|f| > C\}) = \mu(\cup_{n\in\mathbb{N}}\{|f| \geq C + 1/n\}) \leq \sum_{n\in\mathbb{N}} \mu(\{|f| \geq C+1/n\}) = 0$$

we conclude $\|f\|_{L^\infty} \leq C$. □

(b) ★ Show that if $\mu(\Omega) < +\infty$, then for any f as in part (a) we have that $\|f\|_{L^\infty} = \lim\limits_{p\to\infty} \|f\|_{L^p}$.

Solution: Choose a sequence $(p_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\|f\|_{L^{p_k}} = \liminf_{p\to\infty}\|f\|_{L^p}$ and let $\varepsilon > 0$. Take k_0, such that $\|f\|_{L^{p_k}} \leq \liminf_{p\to\infty}\|f\|_{L^p} + \varepsilon$ for $k \geq k_0$. Analogous to (a), it follows $\|f\|_{L^\infty} \leq \liminf_{p\to\infty}\|f\|_{L^p} + \varepsilon$ and by letting $\varepsilon \downarrow 0$, we deduce $\|f\|_{L^\infty} \leq \liminf_{p\to\infty}\|f\|_{L^p}$.

For the opposite bound, choose a sequence $(p_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\|f\|_{L^{p_k}} = \limsup_{p\to\infty}\|f\|_{L^p}$. For $q > p$, we have $\|f\|_{L^q}^q \leq \|f\|_{L^p}^p\|f\|_{L^\infty}^{q-p}$. Take $p > 1$ and $k_0 \in \mathbb{N}$, such that $p_k > p$ for $k \geq k_0$. It follows

$$\|f\|_{L^{p_k}} \leq \|f\|_{L^p}^{\frac{p}{p_k}}\|f\|_{L^\infty}^{1-\frac{p}{p_k}} \xrightarrow{k\to\infty} 1\cdot\|f\|_{L^\infty}.$$

As a result, we see $\limsup_{p\to\infty}\|f\|_{L^p} = \lim_{k\to\infty}\|f\|_{L^{p_k}} \leq \|f\|_{L^\infty}$. Thus the limit is established. □

Exercise 12.7.

Let $(x_{n,m})_{(n,m)\in\mathbb{N}^2} \subset [0,+\infty]$ be a sequence parametrized by $\mathbb{N}^2$. Show that

$$\sum_{(n,m)\in\mathbb{N}^2} x_{n,m} = \sum_{n=0}^\infty\sum_{m=0}^\infty x_{n,m} = \sum_{m=0}^\infty\sum_{n=0}^\infty x_{n,m}.$$

Remark. Given a sequence $(x_\alpha)_{\alpha\in A} \subset [0,+\infty]$ parametrized by an arbitrary set A, we define

$$\sum_{\alpha\in A} x_\alpha := \sup_{F\subset A \text{ finite}} \sum_{\alpha\in F} x_\alpha.$$

Solution: We show that $\sum_{(n,m)\in\mathbb{N}^2} x_{n,m} = \sum_{n=0}^\infty\sum_{m=0}^\infty x_{n,m}$, then the other equality follows analogously. Let $F \subset \mathbb{N}^2$ be any finite set, then there exists $N \in \mathbb{N}$ such that $F \subset \{0,1,\ldots,N\} \times \{0,1,\ldots,N\}$. Hence we get that

$$\sum_{(n,m)\in F} x_{n,m} \leq \sum_{n=0}^N\sum_{m=0}^N x_{n,m} \leq \sum_{n=0}^\infty\sum_{m=0}^\infty x_{n,m}.$$

Taking the supremum over all $F \subset \mathbb{N}^2$, we thus get that $\sum_{(n,m)\in\mathbb{N}^2} x_{n,m} \leq \sum_{n=0}^\infty\sum_{m=0}^\infty x_{n,m}$. Let us now prove the reversed inequality. It is sufficient to show that $\sum_{n=0}^N\sum_{m=0}^\infty x_{n,m} \leq \sum_{(n,m)\in\mathbb{N}^2} x_{n,m}$ for all $N \in \mathbb{N}$. Note that

$$\sum_{n=0}^N\sum_{m=0}^\infty x_{n,m} = \lim_{M\to\infty}\sum_{n=0}^N\sum_{m=0}^M x_{n,m} = \lim_{M\to\infty}\sum_{(n,m)\in\{0,\ldots,N\}\times\{0,\ldots,M\}} x_{n,m} \leq \sum_{(n,m)\in\mathbb{N}^2} x_{n,m},$$

which concludes the proof. □