Exercise 13.1. ♣

(a) The value of the limit

$$\lim_{k\to\infty}\int_0^k \left(1-\frac{x}{k}\right)^k e^{x/3}\,dx$$

is

(A) 0. (B) $\frac{3}{2}$. ✓ (C) 1. (D) $+\infty$.

Solution: This follows from Dominated Convergence by using the dominating function $e^{-2x/3}$, which is summable on $(0,\infty)$. More precisely, we need to show that

$$\left(1-\frac{x}{k}\right)^k \chi_{[0,k]}(x) \le e^{-x}.$$

For $k < x$ this is trivial; for $k \ge x$, since $1 - x/k > 0$, we may take both sides to the power $1/k$ and use the change of variable $y = x/k$, so the inequality to prove becomes $1 - y \le e^{-y}$, which is well known. Then we can pass to the limit under the integral and obtain

$$\lim_{k\to\infty}\int_0^k \left(1-\frac{x}{k}\right)^k e^{x/3}\,dx = \int_0^\infty \lim_{k\to\infty}\left(1-\frac{x}{k}\right)^k e^{x/3}\,dx = \int_0^\infty e^{-x}\cdot e^{x/3}\,dx = \int_0^\infty e^{-2x/3}\,dx = \frac{3}{2}.$$

(b) The value of the limit

$$\lim_{n\to\infty} n\int_0^\infty \frac{\sin\left(\frac{x}{n}\right)}{x(1+x^2)}\,dx$$

is

(A) 0. (B) $\frac{\pi}{2}$. ✓ (C) $\frac{\pi}{4}$. (D) 1.

Solution: Write the integrals as

$$\int_0^\infty f_n(x)\frac{1}{1+x^2}\,dx$$

where we have used

$$f_n(x) := \frac{\sin(x/n)}{x/n},$$

and notice that $|f_n(x)| \le 1$ for every $x \in (0,\infty)$. Moreover observe that $f_n(x) \to 1$ as $n \to \infty$ for every $x \in (0,\infty)$. Hence we may apply Dominated Convergence and deduce

$$\lim_{n\to\infty}\int_0^\infty f_n(x)\frac{1}{1+x^2}\,dx = \int_0^\infty 1\frac{1}{1+x^2}\,dx = \arctan(\infty) - \arctan(0) = \frac{\pi}{2}.$$

(c) Let $f_n \in L^1(0,1) \cap L^2(0,1)$ for $n = 1, 2, 3, \ldots$ and consider the following statements:

(i) If $\|f_n\|_{L^1} \to 0$, then $\|f_n\|_{L^2} \to 0$.

(ii) If $\|f_n\|_{L^2} \to 0$, then $\|f_n\|_{L^1} \to 0$.

Which of them are true?

(A) Both (i) and (ii).

(B) (i) but not (ii).
(C) (ii) but not (i). ✓
(D) Neither (i) nor (ii).

Solution: A counterexample to (i) is given by $f_n = n\chi_{(0,1/n^2)}$, which converges to zero in L^1 but has L^2 norm $\|f_n\|_{L^2} \equiv 1$ for all n. On the other hand, (ii) is true thanks to the Hölder inequality.

(d) Is it true that a sequence of functions in $L^1(0,1)$ converging in measure also converges in the L^1 norm?

Solution: No, consider for example $f_n = n\chi_{(0,1/n)}$, which converge to zero in measure but whose L^1 norm is 1 for all n.

Exercise 13.2.
Consider the functions
$$f_n(x) = \sqrt{n}\chi_{[\log(n),\log(n+1)]}(x)$$
defined on $(0,\infty)$. Determine the values of $p \in [1,+\infty]$ such that $f_n \to 0$ in L^p as $n \to \infty$.

Solution: The suprema of these functions clearly diverges, so it is enough to consider $p < \infty$. We compute the norm of f_n:
$$\|f_n\|_{L^p} = \left(\int_0^\infty \sqrt{n}^p \chi_{[\log(n),\log(n+1)]}\right)^{1/p} = \left(n^{p/2} \cdot \log\frac{n+1}{n}\right)^{1/p}$$
Recall that $\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) = 1$. Thus we may rearrange the above expression as
$$\|f_n\|_{L^p} = \left(n^{p/2-1} \cdot n\log\frac{n+1}{n}\right)^{1/p}.$$
Now it is clear that for $p < 2$ the first factor goes to zero, and for $p \geq 2$ it is bounded below, while the second factor converges to 1. Therefore we have convergence in L^p precisely for $p \in [1,2)$. $\square$

Exercise 13.3.
Let $f \in L^p(\mathbb{R},\lambda)$, where λ is the Lebesgue measure. By means of Fubini's Theorem, show that the following equality holds:
$$\int_{\mathbb{R}} |f(x)|^p dx = p\int_0^\infty y^{p-1}\lambda(\{x \in \mathbb{R} : |f(x)| \geq y\})\, dy.$$

Hint: $|f(x)|^p = \int_0^{|f(x)|} py^{p-1} dy$.

Solution: It is easy to see that $|f(x)|^p = \int_0^{|f(x)|} py^{p-1}dy$. Therefore, using Fubini's Theorem in the second line (to change the order of integration), we get

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^p dx &= \int_{\mathbb{R}} \left(\int_0^{|f(x)|} py^{p-1}dy \right) dx = p \int_{\mathbb{R}} \left(\int_{\mathbb{R}} y^{p-1} \chi_{[0,|f(x)|]}(y) dy \right) dx \\ &= p \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_{\{(x,y)\in\mathbb{R}^2 :\, 0 \le y \le |f(x)|\}}(x,y)\, dx \right) y^{p-1} dy \\ &= p \int_{\mathbb{R}} \lambda(\{x \in \mathbb{R} : |f(x)| \ge y\}) \chi_{[0,+\infty)}(y) y^{p-1} dy \\ &= p \int_0^{\infty} y^{p-1} \lambda(\{x \in \mathbb{R} : |f(x)| \ge y\}) dy. \end{aligned}$$

□

Exercise 13.4.
Define the function $f : [0,1]^2 \to \mathbb{R}$ as

$$f(x,y) := \begin{cases} y^{-2} & \text{if } 0 < x < y < 1, \\ -x^{-2} & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Is this function summable with respect to the Lebesgue measure?

Solution: We want to prove that f is not summable. Suppose it were summable. Then, we could change the order of integration thanks to Fubini's Theorem. However, this leads to a contradiction since

$$\int_0^1 \int_0^1 f(x,y) dx dy = \int_0^1 \left(\int_0^y \frac{1}{y^2} dx - \int_y^1 \frac{1}{x^2} dx \right) dy = 1$$

and

$$\int_0^1 \int_0^1 f(x,y) dy dx = \int_0^1 \left(\int_0^x -\frac{1}{x^2} dy + \int_x^1 \frac{1}{y^2} dy \right) dx = -1 \ .$$

□

Exercise 13.5.
Let $1 \le p < +\infty$ and $f \in L^p(\mathbb{R}^n)$ and, for all $h \in \mathbb{R}^n$, consider the function $\tau_h \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\tau_h(x) = x + h$. Show that

$$\|f \circ \tau_h - f\|_{L^p} \to 0 \qquad \text{as } h \to 0.$$

Hint: use the density of continuous and compactly supported functions in L^p (Theorem 3.7.15 in the Lecture Notes).

Solution: Fix $\varepsilon > 0$, then by Theorem 3.7.15 there exists $g \in C_c^0(\mathbb{R}^n)$ such that $\|f - g\|_{L^p} < \varepsilon/3$. Define the compact set $K = \{x \in \mathbb{R}^n \mid d(x, \operatorname{supp}(g)) \leq 1\}$, then for $|h| \leq 1$ we have

$$\|g \circ \tau_h - g\|_{L^p}^p = \int_K |g(x+h) - g(x)|^p dx \leq \mathcal{L}^n(K) \sup_{|x-y| \leq h} |g(x) - g(y)|.$$

Therefore, using that g is uniformly continuous, there exists $r > 0$ such that $\|g \circ \tau_h - g\|_{L^p} < \varepsilon/3$ for all $|h| \leq r$. Hence, for all $|h| \leq r$, we have

$$\|f \circ \tau_h - f\|_{L^p} \leq \|f \circ \tau_h - g \circ \tau_h\|_{L^p} + \|g \circ \tau_h - g\|_{L^p} + \|g - f\|_{L^p} < \varepsilon,$$

which proves what we wanted by arbitrariness of ε. □

Exercise 13.6. ★
We say that a family $(\varphi_\varepsilon)_{\varepsilon>0}$ of functions in $L^1(\mathbb{R}^n)$ is an *approximate identity* if:

1. $\varphi_\varepsilon \geq 0$ and $\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = 1$ for all $\varepsilon > 0$;
2. for all $\delta > 0$ we have that $\int_{\{|x| \geq \delta\}} \varphi_\varepsilon(x) dx \to 0$ as $\varepsilon \to 0$.

(a) Given $\varphi \in L^1(\mathbb{R}^n)$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi(x) dx = 1$, define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ for all $\varepsilon > 0$. Show that $(\varphi_\varepsilon)_{\varepsilon>0}$ is an approximate identity.
Solution: Obviously we have that $\varphi_\varepsilon \geq 0$. Moreover

$$\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = \int_{\mathbb{R}^n} \varphi(\varepsilon^{-1}(x))\varepsilon^{-n} dx = \int_{\mathbb{R}^n} \varphi(y) dy = 1,$$

where we made the change of variable $y = \varepsilon^{-1}x$ and we used the fact that $\mathcal{L}^n(\varepsilon^{-1}A) = \varepsilon^{-n}\mathcal{L}^n(A)$ for all $\mathcal{L}^n$-measurable sets A. Fix now $\delta > 0$, using the same change of variable we get

$$\int_{\{|x| \geq \delta\}} \varphi_\varepsilon(x) dx = \int_{\{|x| \geq \delta\}} \varphi(\varepsilon^{-1}(x))\varepsilon^{-n} dx = \int_{\{|y| \geq \varepsilon^{-1}\delta\}} \varphi(y) dy,$$

which converges to 0 by the Dominated Convergence Theorem, since the functions $\varphi\chi_{\{|y| \geq \varepsilon^{-1}\delta\}}$ converge pointwise to zero almost everywhere and are dominated by the $\mathcal{L}^n$-summable function φ. □

Let $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^n)$ be an approximate identity. Show that the following statements hold.
(b) If $f \in L^\infty(\mathbb{R}^n)$ is continuous at $x_0 \in \mathbb{R}^n$, then $f * \varphi_\varepsilon$ is continuous and $(f * \varphi_\varepsilon)(x_0) \to f(x_0)$ as $\varepsilon \to 0^+$.
Solution: Let us first prove that $f * \varphi_\varepsilon$ is continuous. Note that, for all $h \in \mathbb{R}^n$, we have

$$(f * \varphi_\varepsilon)(x + h) = \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x + h - y) dy = \int_{\mathbb{R}^n} f(y)(\varphi_\varepsilon \circ \tau_h)(x - y) dy = (f * (\varphi_\varepsilon \circ \tau_h))(x).$$

Hence, applying Corollary 4.4.6 (ii) to the functions $f \in L^\infty(\mathbb{R}^n)$ and $\varphi_\varepsilon \circ \tau_h - \varphi_\varepsilon \in L^1(\mathbb{R}^n)$, we get

$$|(f * \varphi_\varepsilon)(x + h) - (f * \varphi_\varepsilon)(x)| = |(f * (\varphi_\varepsilon \circ \tau_h - \varphi_\varepsilon))(x)| \leq \|f\|_{L^\infty} \|\varphi_\varepsilon \circ \tau_h - \varphi_\varepsilon\|_{L^1},$$

which converges to 0 as $h \to 0$ thanks to Exercise 13.5. This proves that $f * \varphi_\varepsilon$ is continuous.

Given $\delta > 0$, by continuity of f at x_0, there exists $r > 0$ such that $|f(x_0 - y) - f(x_0)| < \delta$ for all $|y| < r$. Hence, using that $\int_{\mathbb{R}^n} \varphi_\varepsilon = 1$, we get

$$\begin{aligned}
|(f * \varphi_\varepsilon)(x_0) - f(x_0)| &\le \int_{\mathbb{R}^n} |f(x_0 - y) - f(x_0)|\varphi_\varepsilon(y)dy \\
&= \int_{\{|y|<r\}} |f(x_0 - y) - f(x_0)|\varphi_\varepsilon(y)dy + \int_{\{|y|\ge r\}} |f(x_0 - y) - f(x_0)|\varphi_\varepsilon(y)dy \\
&\le \delta + 2\|f\|_{L^\infty} \int_{\{|y|\ge r\}} \varphi_\varepsilon(y)dy.
\end{aligned}$$

The RHS can be made smaller than 2δ by the second property of an approximation of the identity. This concludes the proof by arbitrariness of δ. □

(c) If $f \in L^\infty(\mathbb{R}^n)$ is uniformly continuous, then $f * \varphi_\varepsilon \xrightarrow{L^\infty} f$ as $\varepsilon \to 0^+$.

Solution: The solution works the same as the one of part (b) using that, given $\delta > 0$, there exists $r > 0$ such that $|f(x - y) - f(x)| < \delta$ for all $|y| < r$, where r does not depend on x. □

(d) If $1 \le p < +\infty$ and $f \in L^p(\mathbb{R}^n)$, then $f * \varphi_\varepsilon \xrightarrow{L^p} f$ as $\varepsilon \to 0^+$.

Hint: use Hölder's inequality and keep in mind Exercise 13.5 and part (b).

Solution: First note that, by Corollary 4.4.6 (ii), $f * \varphi_\varepsilon \in L^p(\mathbb{R}^n)$. Now, using that $\int_{\mathbb{R}^n} \varphi_\varepsilon = 1$ and Hölder inequality, we get

$$\begin{aligned}
|(f * \varphi_\varepsilon)(x) - f(x)|^p &\le \left| \int_{\mathbb{R}^n} (f(x - y) - f(x))\varphi_\varepsilon(y)dy \right|^p \\
&= \left| \int_{\mathbb{R}^n} (f(x - y) - f(x))\varphi_\varepsilon(y)^{1/p}\varphi_\varepsilon(y)^{1/p'}dy \right|^p \\
&\le \left(\int_{\mathbb{R}^n} |f(x - y) - f(x)|^p \varphi_\varepsilon(y)dy \right) \left(\int_{\mathbb{R}^n} \varphi_\varepsilon(y)dy \right)^{p/p'} \\
&= \int_{\mathbb{R}^n} |f(x - y) - f(x)|^p \varphi_\varepsilon(y)dy.
\end{aligned}$$

Then we integrate over $\mathbb{R}^n$ and use Tonelli's theorem to get

$$\begin{aligned}
\int_{\mathbb{R}^n} |(f * \varphi_\varepsilon)(x) - f(x)|^p dx &\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y) - f(x)|^p \varphi_\varepsilon(y)dydx \\
&= \int_{\mathbb{R}^n} \varphi_\varepsilon(y) \left(\int_{\mathbb{R}^n} |f(x - y) - f(x)|^p dx \right) dy = \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\|f \circ \tau_{-y} - f\|_{L^p}^p dy.
\end{aligned}$$

Now denote by $g\colon \mathbb{R}^n \to [0, +\infty)$ the function $g(y) = \|f \circ \tau_{-y} - f\|_{L^p}^p$. Observe that, by Exercise 13.5, the function g is continuous at 0. Moreover $g(y) \le 2^p\|f\|_{L^p}^p$, hence $g \in L^\infty(\mathbb{R}^n)$. Therefore we can use part (b) to obtain that $(g * \varphi_\varepsilon)(0) \to g(0) = 0$ as $\varepsilon \to 0$. However note that this concludes the proof since $\int_{\mathbb{R}^n} \varphi_\varepsilon(y)\|f \circ \tau_{-y} - f\|_{L^p}^p dy = (g * \varphi_\varepsilon)(0)$. □

Exercise 13.7.

Compute the following limits:

(a)

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx}{(1+x)^n}\,dx.$$

Solution: It is clear that the constant function 1, which is summable on $[0,1]$, dominates the sequence. Moreover, for all $x>0$ the integrand tends to 0 as $n\to\infty$. Therefore, by the dominated convergence theorem,

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx}{(1+x)^n}\,dx = \int_0^1 \lim_{n\to\infty}\frac{1+nx}{(1+x)^n}\,dx = \int_0^1 0\,dx = 0. \qquad \square$$

(b)

$$\lim_{n\to\infty}\int_0^1 \frac{x\log x}{1+n^2x^2}\,dx.$$

Solution: The integrand is clearly bounded above by the function $x|\log x|$, which is bounded on $(0,1)$ and therefore summable. Moreover, the sequence of integrands tends to 0 away from $x=0$. Therefore, as above, the limit of the integrals is 0. $\square$

Exercise 13.8.
Let $I=[0,1]$ and consider the function

$$f: I^3\to[0,\infty],\quad f(x,y,z):=\begin{cases}\frac{1}{\sqrt{|y-z|}}, & \text{if } y\neq z,\\ \infty, & \text{if } y=z.\end{cases}$$

Show that $f\in L^1(I^3,\mathcal{L}^3)$.

Solution: Note that $f\geq 0$ and that f is continuous outside the closed set $\{y=z\}$. This shows that f is Lebesgue-measurable. This allows us to apply Tonelli's theorem twice:

$$\begin{aligned}\int_{I^3} f(x,y,z)d\mathcal{L}^3(x,y,z) &= \int_I\left(\int_{I^2} f(x,y,z)d\mathcal{L}^2(y,z)\right)d\mathcal{L}^1(x)\\ &= \int_I\left(\int_I\left(\int_I f(x,y,z)d\mathcal{L}^1(y)\right)d\mathcal{L}^1(z)\right)d\mathcal{L}^1(x).\end{aligned}$$

Now we compute the inner integral for x,z fixed:

$$\begin{aligned}\int_I f(x,y,z)\,d\mathcal{L}^1(y) &= \int_{I\setminus\{z\}} \frac{1}{\sqrt{|y-z|}}\,d\mathcal{L}^1(y)\\ &= \int_0^z \frac{1}{\sqrt{z-y}}\,d\mathcal{L}^1(y) + \int_z^1 \frac{1}{\sqrt{y-z}}\,d\mathcal{L}^1(y)\\ &= \left[-2\sqrt{z-y}\right]_{y=0}^{y=z} + \left[2\sqrt{y-z}\right]_{y=z}^{y=1}\\ &= 2\sqrt{z}+2\sqrt{1-z}.\end{aligned}$$

Therefore for each $x \in I$ we have

$$\int_{I^2} f(x,y,z)\, d\mathcal{L}^2(y,z) = \int_I 2\sqrt{z} + 2\sqrt{1-z}\, d\mathcal{L}^1(z) = \frac{8}{3},$$

and finally we get

$$\int_{I^3} |f(x,y,z)|\, d\mathcal{L}^3(x,y,z) = \int_{I^3} f(x,y,z)\, d\mathcal{L}^3(x,y,z) = \int_I \frac{8}{3}\, d\mathcal{L}^1(x) = \frac{8}{3} < \infty,$$

which shows that $f \in L^1(I^3, \mathcal{L}^3)$. □

Exercise 13.9. ★

The goal of this exercise is to construct an $\mathcal{L}^1$-measurable set $A \subset [0,1]$ with the property that both

$$\mathcal{L}^1(U \cap A) > 0 \quad \text{and} \quad \mathcal{L}^1(U \setminus A) > 0 \tag{$*$}$$

for every nonempty open subset $U \subset [0,1]$.

(a) Show that it is enough to check $(*)$ for dyadic intervals U, that is, for sets U of the form $U = (m2^{-j}, (m+1)2^{-j})$ with integers $j \geq 1$ and $0 \leq m < 2^j$.

Solution: Let U be an arbitrary nonempty open set in $[0,1]$. Then it contains a dyadic interval $I = (m2^{-j}, (m+1)2^{-j})$, so that

$$\mathcal{L}^1(U \cap A) \geq \mathcal{L}^1(I \cap A) > 0 \quad \text{and} \quad \mathcal{L}^1(U \setminus A) \geq \mathcal{L}^1(I \setminus A) > 0.$$ □

The idea of the proof will be to modify iteratively our set by small amounts, so that its measure in all dyadic intervals of smaller and smaller sizes is controlled from above and below. This is the main construction that we will need in the iteration:

(b) Show that given any measurable set $E \subset [0,1]$, any integer $k \geq 1$ and any real number $0 < \beta \leq 2^{-(k+1)}$, one can find a measurable set $B \subset [0,1]$ such that

$$\mathcal{L}^1((m2^{-k}, (m+1)2^{-k}) \cap B) \geq \beta \text{ and } \mathcal{L}^1((m2^{-k}, (m+1)2^{-k}) \setminus B) \geq \beta \tag{1}$$

for $m = 0, 1, \ldots, 2^k - 1$, and

$$\mathcal{L}^1(E \triangle B) \leq 2^k \beta. \tag{2}$$

Solution: We will modify the set E in each of the intervals $I_m := (m2^{-k}, (m+1)2^{-k})$ by adding or subtracting to it a set of measure at most β, depending on whether $\lambda_m := \mathcal{L}^1(I_m \cap E)$ is too large or too small.

More precisely, for each $m \in \{0, 1, \ldots, 2^k - 1\}$, we construct a set $B_m \subset I_m$ with $\mathcal{L}^1(B_m \triangle (E \cap I_m)) \leq \beta$ satisfying (1) for the corresponding m. In order to do that, we distinguish three cases:

- If $\mathcal{L}^1(I_m \setminus E) = 2^{-k} - \lambda_m < \beta$ ("E is too large"), this means that $2^{-k} - \beta < \lambda_m$, so using Exercise 3.1 we can find a measurable set $B_m \subset I_m \cap E$ with $\mathcal{L}^1(B_m) = 2^{-k} - \beta$. Then $\mathcal{L}^1(I_m \setminus B_m) = \beta$, $\mathcal{L}^1(I_m \cap B_m) = 2^{-k} - \beta \geq \beta$, and also

$$\begin{aligned}\mathcal{L}^1(I_m \cap (E \triangle B_m)) &= \mathcal{L}^1(I_m \cap E \setminus B_m) = \mathcal{L}^1(I_m \cap E) - \mathcal{L}^1(B_m) \\ &= \lambda_m - (2^{-k} - \beta) = \beta - (2^{-k} - \lambda_m) \leq \beta.\end{aligned}$$

- If $\mathcal{L}^1(I_m \cap E) < \beta$ ("E is too small") we argue similarly but with the complement of E.
- If none of the above strict inequalities holds, then we can take $B_m = I_m \cap E$.

Finally choosing $B := B_0 \cup \cdots \cup B_{2^k-1}$ the two properties (1) and (2) are easily satisfied. □

Let us now fix a sequence of positive real numbers $\beta_1, \beta_2, \ldots$ satisfying the following condition:

$$\forall k \geq 1 \qquad 2^{-(k+1)} \geq \beta_k > 2^{k+1}\beta_{k+1} + 2^{k+2}\beta_{k+2} + 2^{k+3}\beta_{k+3} + \cdots . \tag{C}$$

We construct inductively using part (b) a sequence of measurable sets $A_0, A_1, A_2, \ldots \subset [0,1]$ with $A_0 = \varnothing$ satisfying the following two properties:

$$\mathcal{L}^1((m2^{-k}, (m+1)2^{-k}) \cap A_k) \geq \beta_k \quad \text{and} \quad \mathcal{L}^1((m2^{-k}, (m+1)2^{-k}) \setminus A_k) \geq \beta_k$$

for $m = 0, 1, \ldots, 2^k - 1$, and

$$\mathcal{L}^1(A_{k-1} \triangle A_k) \leq 2^k \beta_k.$$

(c) Show that there exists a measurable set $A \subset [0,1]$ such that $\mathcal{L}^1(A_k \triangle A) \to 0$ as $k \to \infty$.
Hint: Use the completeness of L^1.

Solution: Let $f_k := \chi_{A_k}$ and observe that $\|f_j - f_k\|_{L^1} = \mathcal{L}^1(A_j \triangle A_k)$. We claim that this is a Cauchy sequence: for $j < k$,

$$\begin{aligned}\|f_j - f_k\|_{L^1} &\leq \|f_j - f_{j+1}\|_{L^1} + \|f_{j+1} - f_{j+2}\|_{L^1} + \cdots + \|f_{k-1} - f_k\|_{L^1} \\ &= \mathcal{L}^1(A_j \triangle A_{j+1}) + \mathcal{L}^1(A_{j+1} \triangle A_{j+2}) + \cdots + \mathcal{L}^1(A_{k-1} \triangle A_k) \\ &\leq 2^{j+1}\beta_{j+1} + 2^{j+2}\beta_{j+2} + \cdots + 2^k \beta_k \\ &\leq \sum_{\ell=j+1}^{\infty} 2^\ell \beta_\ell \xrightarrow{j\to\infty} 0\end{aligned}$$

because the sum is finite, thanks to condition (C) with $k = 1$. Thus $\{f_k\}$ converges in L^1 to a measurable function f by the completeness of $L^1([0,1])$. In particular, a subsequence f_{k_j} converges pointwise to f almost everywhere, thus f can only take the values 0 and 1 and hence can be written as $f = \chi_A$ for a measurable set A. Finally

$$\mathcal{L}^1(A_k \triangle A) = \|f_k - f\|_{L^1} \xrightarrow{k\to\infty} 0.$$ □

(d) Show that $(*)$ holds for this set A and any dyadic interval U.

Solution: Let $U = (m2^{-k}, (m+1)2^{-k})$ and consider $j > k$. Then

$$\begin{aligned}\mathcal{L}^1(U \cap A_j) &\geq \mathcal{L}^1(U \cap A_k) - \mathcal{L}^1(A_k \triangle A_j) \\ &\geq \mathcal{L}^1(U \cap A_k) - \big(\mathcal{L}^1(A_k \triangle A_{k+1}) + \cdots + \mathcal{L}^1(A_{j-1} \triangle A_j)\big) \\ &\geq \beta_k - \left(2^{k+1}\beta_{k+1} + \cdots + 2^j \beta_j\right).\end{aligned}$$

(These computations are just the application of the triangle inequality to χ_{A_j}.) Letting $j \to \infty$ the left hand side converges to $\mathcal{L}^1(U \cap A)$, so using condition (C) we get

$$\mathcal{L}^1(U \cap A) \geq \beta_k - \left(2^{k+1}\beta_{k+1} + 2^{k+2}\beta_{k+2} + \cdots\right) > 0.$$

The same argument applied to the complement of A shows the second inequality. □

(e) To complete the proof, show that if we choose $\beta_k = 2^{-3^k}$, then condition (C) holds.

Solution: Clearly $2^{-3^k} \leq 2^{-(k+1)}$ for $k \geq 1$. For the second inequality of (C), we claim that $2^{k+j} \cdot 2^{-3^{k+j}} < 2^{-3^k - j}$. This shows the inequality:

$$\sum_{j=1}^{\infty} 2^{k+j} \cdot 2^{-3^{k+j}} < \sum_{j=1}^{\infty} 2^{-3^k - j} = 2^{-3^k} \sum_{j=1}^{\infty} 2^{-j} = 2^{-3^k}.$$

The claim follows by a simple argument: $k + j - 3^{k+j} < -3^k - j \iff k + 2j + 3^k < 3^{k+j}$ clearly holds for $j = 1$ and every $k \geq 1$, and for higher values of j it follows by induction. □