**1. Multiple choice** There is only one correct answer in each question.

**(a)** In which open set the following power series defines a holomorphic function?

$$\sum_{n=0}^{+\infty} \frac{e^{in} i^{3n}}{2n^2 + 1/n!} z^n.$$

- ◯ $\{z \in \mathbb{C} : -1/2 < \Im(z) < 1/2\}$.
- ◯ $\{z \in \mathbb{C} : |z| < 1\}$.
- ◯ $\mathbb{C}$.
- ◯ $\{z \in \mathbb{C} : 1 < |z| < 2\}$

**(b)** Which of the following functions is *not* meromorphic?

- ◯ $\sin(z)$.
- ◯ $\frac{1}{\sin(z)}$.
- ◯ $\frac{z^3}{z^4+1}$.
- ◯ $\sin(1/z)$.

**(c)** Which $\Omega \subset \mathbb{C}$ is *not* biholomorphic to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$?

- ◯ $\Omega = \{z \in \mathbb{C} : \Re(z) > 0\}$.
- ◯ $\Omega = \{z \in \mathbb{C} : |z| < \Re(z)^2 + 1\}$.
- ◯ $\Omega = \mathbb{C} \setminus \{0\}$.
- ◯ $\Omega = \mathbb{C}$.

**(d)** Consider the singularity of $f(z) = \frac{\sin(z)\cos(1/z)}{(\pi - z)^{2023}}$ in $z = z_0 = \pi$. Then, $z_0$ is

- ◯ a pole of order 2022.
- ◯ a removable singularity.
- ◯ a pole of order 2023.
- ◯ an essential singularity.

**(e)** Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and $f : \mathbb{D} \to \mathbb{C}$ holomorphic. Which assertion does *not* imply that $f$ is constant?

- ◯ $|f(z)| \leq |f(i/4)|$ for all $z \in \mathbb{D}$.
- ◯ $f(0) = 0,\ f'(0) = 0$.
- ◯ $f(1/(2n)) = 1$, for all $n \in \mathbb{N}$.
- ◯ $\int_{\{|z|=1/2\}} \frac{f(z)}{z^k}\, dz = 0$ for all $k \in \mathbb{N}$.

**(f)** Let log be the principal branch of the logarithm, and $\gamma$ the positively oriented arc $\{e^{it} : t \in [0, \pi/2]\}$. What is the value of

$$\int_\gamma \log(z^2)\, dz$$

equal to?

○ $2i$.
○ $\pi + 2 - i$.
○ $\pi - 2 + 2i$.
○ $2 - 2i - \pi$.

**(g)** How many zeros has the polynomial $p(z) = z^5 + 5z - \pi$ inside the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$?

○ 2.
○ 4.
○ 3.
○ 5.

**In the following exercises, please justify all steps.**

**2.** Consider the meromorphic function

$$f(z) = \frac{\sin(\pi z)}{z(z^2+1)}.$$

**(a)** Find the zeros of $f$ and their order.

**(b)** Find the poles of $f$ and their order.

**(c)** Compute the integral

$$\int_\gamma f\, dz,$$

when $\gamma$ is the circle of radius 3 centered in $i$ positively oriented.

**3.** Compute the following real integral

$$\int_{-\infty}^{+\infty} \frac{\cos(\sqrt{2}t)}{t^4+1}\, dt.$$

*Hint: Write this as a complex integral, and consider a contour parametrizing the boundary of a half disc.*

**4.** Let $f : \mathbb{C} \to \mathbb{C}$ holomorphic and injective, with $f(0) = 0$.

**(a)** Show that for every $r > 0$ there exists $\varepsilon > 0$ such that $|f(z)| > \varepsilon$ for every $z \in \mathbb{C}$ satisfying $|z| \geq r$.

**(b)** Show that the singularity at zero of the function

$$g : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad g(z) := f\left(\frac{1}{z}\right),$$

is a pole.

**(c)** Conclude that $f$ is a complex polynomial, and therefore $f(z) = cz$ for some $c \in \mathbb{C} \setminus \{0\}$.

*Hints: For part (a) take advantage of the Open Mapping Theorem. For part (b) take advantage of the Casorati-Weierstrass and Liouville Theorems.*

**5.** Let $\Omega \subset \mathbb{C}$ be an open and connected set containing the origin, and $f : \Omega \setminus \{0\} \to \mathbb{C}$ holomorphic. Suppose that there exists a sequence $(z_n)$ in $\Omega$ such that $\lim_{n\to+\infty} z_n = 0$ and

$$|f(z_n)| \leq e^{-1/|z_n|},$$

for all $n \in \mathbb{N}$.

**(a)** Prove that $f$ has a removable singularity in zero if and only if $f$ is constantly equal to zero in $\Omega$.

**(b)** Deduce that $f$ is either a constant, or it has an essential singularity in zero.