1. Multiple choice Only one correct answer each question.

(a) In which open set is the following power series defining an holomorphic function?

$$\sum_{n=0}^{+\infty} \frac{e^{in} i^{3n}}{2n^2 + 1/n!} z^n.$$

- ○ $\{z \in \mathbb{C} : -1/2 < \Im(z) < 1/2\}$.
- ● $\{z \in \mathbb{C} : |z| < 1\}$.
- ○ $\mathbb{C}$.
- ○ $\{z \in \mathbb{C} : 1 < |z| < 2\}$

SOL: We compute the radius of convergence ρ of the power series as

$$\rho^{-1} = \lim_{n\to+\infty} \left| \frac{2n^2 + 1/n!}{2(n+1)^2 + 1/(n+1)!} \right| = \lim_{n\to+\infty} \left| \frac{2n^2}{2(n+1)^2} \right| = 1.$$

(b) Which of the following functions is *not* meromorphic?

- ○ $\sin(z)$.
- ○ $\frac{1}{\sin(z)}$.
- ○ $\frac{z^3}{z^4+1}$.
- ● $\sin(1/z)$.

SOL: The point $z = 0$ is an essential singularity of $\sin(1/z)$, and hence this function cannot be meromorphic.

(c) Which $\Omega \subset \mathbb{C}$ it is *not* biholomorphic to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$?

- ○ $\Omega = \{z \in \mathbb{C} : \Re(z) > 0\}$.
- ○ $\Omega = \{z \in \mathbb{C} : |z| < \Re(z)^2 + 1\}$.
- ○ $\Omega = \mathbb{C} \setminus \{0\}$.
- ● $\Omega = \mathbb{C}$.

SOL: This is not possible by Liouville's Theorem: if $f : \mathbb{C} \to \mathbb{D}$ is holomorphic, then $|f(z)| \leq 1$ for all $z \in \mathbb{C}$, and hence f has to be a constant.

(d) Consider the singularity of $f(z) = \frac{\sin(z)\cos(1/z)}{(\pi - z)^{2023}}$ in $z = z_0 = \pi$. Then, z_0 is

- ● a pole of order 2022.
- ○ a removable singularity.
- ○ a pole of order 2023.
- ○ an essential singularity.

SOL: Taking advantage of the Tayolor series of $\sin(z)$ and $\cos(1/z)$ in π we get that

$$\begin{aligned}\frac{\sin(z)\cos(1/z)}{(\pi - z)^{2023}} &= (\pi - z)^{-2023}((z-\pi) + O((z-\pi)^2))(\cos(1/\pi) + O(z-\pi)) \\ &= -\cos(1/\pi)(\pi - z)^{-2022} + O((z-\pi)^{-2021}).\end{aligned}$$

Hence, the pole is of order 2022.

(e) Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and $f : \mathbb{D} \to \mathbb{C}$ holomorphic. Which assertion does *not* imply that f is constant?

- ○ $|f(z)| \leq |f(i/4)|$ for all $z \in \mathbb{D}$.
- ● $f(0) = 0$, $f'(0) = 0$.
- ○ $f(1/(2n)) = 1$, for all $n \in \mathbb{N}$.
- ○ $\int_{\{|z|=1/2\}} \frac{f(z)}{z^k}\, dz = 0$ for all $k \in \mathbb{N}$.

SOL: Consider for instance $f(z) = z^2$.

(f) Let log be the principal branch of the logarithm, and γ the positively oriented arc $\{e^{it} : t \in [0, \pi/2]\}$. What is the value of

$$\int_\gamma \log(z^2)\, dz$$

equal to?

- ○ $2i$.
- ○ $\pi + 2 - i$.
- ○ $\pi - 2 + 2i$.
- ● $2 - 2i - \pi$.

SOL: We compute

$$\int_\gamma \log(z^2)\, dz = \int_0^{\pi/2} \log(e^{2it}) i e^{it}\, dt = -\int_0^{\pi/2} 2te^{it}\, dt = 2 - 2i - \pi.$$

(g) How many zeros has the polynomial $p(z) = z^5 + 5z - \pi$ inside the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$?

- ○ 2.
- ● 4.
- ○ 3.
- ○ 5.

SOL: We apply Rouché Theorem first for $|z| = 1$:

$$|5z| = 5 > \pi + 1 \geq |z^5 - \pi|,$$

and after for $|z| = 2$:

$$|z^5| = 32 > 10 + \pi \geq |5z - \pi|.$$

The number of zeros in the annulus are then $5 - 1 = 4$.

2. Open question Consider the meromorphic function

$$f(z) = \frac{\sin(\pi z)}{z(z^2+1)}.$$

(a) Find the zeros of f and their order.

SOL: The zeros of $\sin(\pi z)$ are exactly $z = k \in \mathbb{Z}$. When $k \neq 0$, $f(k) = 0$, and the order of this zero is one since

$$f'(k) = \frac{\pi\cos(\pi k)k(k^2+1)}{(k(k^2+1))^2} \neq 0.$$

The value $k = 0$ however is not a zero of f, but a removable singularity: in fact notice that

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\pi z + O(z^3)}{z(z^2+1)} = \pi.$$

(b) Find the poles of f and their order.

SOL: As we showed in the previous point, $z = 0$ is not a pole, but a removable singularity. The poles of f are at $z = i$ and $z = -i$. They are of order one since $(z+1) = (z-i)(z+i)$, and the limits $\lim_{z\to i}(z-i)f(z)$ and $\lim_{z\to -i}(z+i)f(z)$ are finite.

(c) Compute the integral

$$\int_\gamma f\, dz,$$

when γ is the circle of radius 3 centered in i positively oriented.

SOL: We first compute the residues of f at i and $-i$:

$$\operatorname{res}_i(f) = \lim_{z\to i}(z-i)f(z) = -\frac{\sin(i\pi)}{2}, \qquad \operatorname{res}_{-i}(f) = \lim_{z\to -i}(z+i)f(z) = \frac{\sin(i\pi)}{2}.$$

Since both poles are in the interior of γ, by the residue theorem

$$\int_\gamma f\, dz = \operatorname{res}_i(f) + \operatorname{res}_{-i}(f) = 0.$$

3. Open question Compute the following real integral

$$\int_{-\infty}^{+\infty} \frac{\cos(\sqrt{2}t)}{t^4+1}\, dt.$$

Hint: Write this as a complex integral, and consider a contour parametrizing the boundary a half disc.

SOL: Let $R > 1$ be fixed, and define γ_R to be the curve parametrizing the boundary of the half disc $\{z \in \mathbb{C} : |z| < R, \Im(z) > 0\}$ with positive orientation, and A_R the arc $\{Re^{i\theta} : \theta \in (0, \pi)\}$. We notice that

$$\int_{-R}^{+R} \frac{\cos(\sqrt{2}t)}{t^4+1}\,dt = \Re\Big(\int_{\gamma_R} \frac{e^{i\sqrt{2}z}}{z^4+1}\,dz - \int_{A_R} \frac{e^{i\sqrt{2}z}}{z^4+1}\,dz\Big).$$

We treat the two integrals in the right hand side separately: first of all notice that

$$\left|\int_{A_R} \frac{e^{i\sqrt{2}z}}{z^4+1}\,dz\right| = \left|\int_0^{\pi} \frac{e^{i\sqrt{2}Re^{i\theta}}Rie^{i\theta}}{R^4e^{4i\theta}+1}\,d\theta\right| \le \frac{R\pi}{(R^4-1)} \max_{\theta\in[0,\pi]} |e^{i\sqrt{2}Re^{i\theta}}|.$$

Notice now that

$$|e^{i\sqrt{2}Re^{i\theta}}| = |e^{i\sqrt{2}R(\cos(\theta)+i\sin(\theta))}| = |e^{-\sqrt{2}R\sin(\theta)}| \le 1,$$

for $\theta \in [0, \pi]$. This proves that

$$\lim_{R\to+\infty}\left|\int_{A_R} \frac{e^{i\sqrt{2}z}}{z^4+1}\,dz\right| \le \lim_{R\to+\infty} \frac{R\pi}{R^4-1} = 0.$$

We compute the second integral via the Residue Theorem: notice that γ_R contains two poles of order one of $f(z) = \frac{e^{i\sqrt{2}z}}{z^4+1}$, namely $(z_1, z_2) = ((1+i)/\sqrt{2}, (-1+i)/\sqrt{2})$. Their residues are

$$\operatorname{res}_{z_1} f = -\frac{(1+i)e^{i-1}}{4\sqrt{2}},$$

and

$$\operatorname{res}_{z_2} f = \frac{(1-i)e^{-i-1}}{4\sqrt{2}}.$$

Hence,

$$\begin{aligned}\int_{\gamma_R} \frac{\cos(\sqrt{2}z)}{z^4+1}\,dz &= 2\pi i\Big(\operatorname{res}_{z_1} f + \operatorname{res}_{z_2} f\Big)\\ &= \frac{\pi i}{2\sqrt{2}}\Big((1-i)e^{-i-1} - (1+i)e^{i-1}\Big)\\ &= \frac{\pi}{\sqrt{2}e}(\sin(1)+\cos(1)).\end{aligned}$$

This proves that

$$\int_{-\infty}^{+\infty} \frac{\cos(\sqrt{2}t)}{t^4+1}\,dt = \lim_{R+\infty} \int_{-R}^{+R} \frac{\cos(\sqrt{2}t)}{t^4+1}\,dt = \Re\Big(\frac{\pi}{\sqrt{2}e}(\sin(1)+\cos(1))\Big)$$
$$= \frac{\pi}{\sqrt{2}e}(\sin(1)+\cos(1)).$$

4. Open question Let $f : \mathbb{C} \to \mathbb{C}$ holomorphic and injective, with $f(0) = 0$.

(a) Show that for every $r > 0$ there exists $\varepsilon > 0$ such that $|f(z)| > \varepsilon$ for every $z \in \mathbb{C}$ satisfying $|z| \geq r$.

SOL: For $\rho > 0$, we denote $B_\rho := \{z \in \mathbb{C} : |z| < \rho\}$. Now, let $r > 0$. By the Open Mapping Theorem $f(B_r)$ is open and contains the origin since $f(0) = 0$. Therefore, there exists a neighbourhood of the origin in the form $B_\varepsilon \subset f(B_r)$ for some $\varepsilon > 0$ small enough. Let now $z \in \mathbb{C} \setminus B_r$, that is $|z| \geq r$. Since f is injective, $f(z) \notin f(B_r)$, so in particular $f(z) \notin B_\varepsilon$, proving that $|f(z)| > \varepsilon$ as wished.

(b) Show that the singularity at zero of the function

$$g : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad g(z) := f\Big(\frac{1}{z}\Big),$$

is a pole.

SOL: We have to prove that zero is neither an essential singularity, nor a removable singularity of f. Suppose by contradiction that zero is an essential singularity of g. Then, by Casorati-Weierstrass for every $\rho > 0$, $g(B_\rho \setminus \{0\})$ is dense in $\mathbb{C}$. This contradicts part (a) taking $r = 1/\rho$ since there exists $\varepsilon > 0$ such that

$$g(B_\rho \setminus \{0\}) = f(\mathbb{C} \setminus B_{1/\rho})$$

does not contain the ball B_ε, and hence it cannot be dense in $\mathbb{C}$. On the other hand, if zero is a removable singularity, then it follows that g extends to a bounded function from $\mathbb{C}$ to $\mathbb{C}$, and hence it must be constant by Liouville's Theorem, contradicting the injectivity of f.

(c) Conclude that f is a complex polynomial, and therefore $f(z) = cz$ for some $c \in \mathbb{C} \setminus \{0\}$.

SOL: Since f is holomorphic, it can be expressed as

$$f(z) = \sum_{n=1}^{+\infty} a_n z^n,$$

in every ball around the origin. Let $k \in \mathbb{N}$ be the order of the pole of g in zero. By definition,

$$g(z) = f(1/z) = \sum_{n=1}^{+\infty} a_n z^{-n}.$$

On the other side, letting $k \in \mathbb{N}$ the order of the pole of g, we have that the coefficients $a_{k+1}, a_{k+2}, \dots$ must be equal to zero, and $a_k \neq 0$, since the principal part of g is of order k. This proves that

$$f(z) = a_1 z + \dots a_k z^k.$$

If $k = 1$ we are done. Otherwise, we have by the Fundamental Theorem of Algebra that there exists $w \in \mathbb{C}$ such that $f(z) = w$ has more than a unique solution, contradicting the injectivity of f.

Hints: For part (a) take advantage of the Open Mapping Theorem. For part (b) take advantage of the Casorati-Weierstrass and Liouville Theorems.

5. Open question Let $\Omega \subset \mathbb{C}$ be an open and connected set containing the origin, and $f : \Omega \setminus \{0\} \to \mathbb{C}$ holomorphic. Suppose that there exists a sequence (z_n) in Ω such that $\lim_{n\to+\infty} z_n = 0$ and

$$|f(z_n)| \leq e^{-1/|z_n|},$$

for all $n \in \mathbb{N}$.

(a) Prove that f has a removable singularity in zero if and only if f is constantly equal to zero in Ω.

SOL: One direction is clear ($f \equiv 0 \Rightarrow f$ has a removable singularity in zero). Let us prove: f has a removable singularity in zero then $f \equiv 0$. By continuity the value of the extension of f in zero is equal to

$$f(0) = \lim_{n\to+\infty} f(z_n) = 0,$$

since $|f(z_n)| \leq e^{-1/|z_n|} \to 0$. If the order of this zero is infinite, then $f \equiv 0$, and we are done. Otherwise, let $k \in \mathbb{N}$ such that $\mathrm{ord}_0 f = k$. Then, by definition, there exists g holomorphic in a neighbourhood $U \subset \Omega$ of zero such that $g(0) \neq 0$ and $f(z) = z^k g(z)$. Let now $N > 0$ big enough so that $z_n \in U$ for all $n \geq N$. Then, from

$$|g(z_n)| = |z_n^{-k} f(z_n)| \leq \frac{e^{-1/|z_n|}}{|z_n|^k}$$

we deduce that $g(0) = \lim_{n\to+\infty} g(z_n) = 0$ since the function $\lim_{t\to 0} e^{-1/|t|}/|t|^k = 0$ for all $k \in \mathbb{N}$. This is a contradiction with the definition of g, proving that $k = \infty$, and hence $f \equiv 0$.

(b) Deduce that f is either a constant, or it has an essential singularity in zero.

SOL: Zero cannot be a pole of f since $\lim_{n\to+\infty}|f(z_n)| = 0 \neq \infty$. Therefore, if it is a removable singularity, then by part (a) $f \equiv 0$. The only option left is when zero is an essential singularity of f.