Exercises with a $\star$ are eligible for bonus points.

1.1. Complex Numbers Review Simplify the following expressions

$$\left(\frac{1-i\sqrt{3}}{2}\right)^{36} =$$

$$\frac{1}{i}\frac{1+2i}{1-2i} - \frac{2+4i}{1+2i} + (1+i)(1-3i) =$$

$$(1+i)^{2n}(1-i)^{2m} = \qquad \text{for every } m, n \in \mathbb{N}.$$

1.2. Power Series Investigate the absolute convergence and radius of convergence of the following power series

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} z^n, \qquad \sum_{n=0}^{+\infty} \frac{e^{in}}{4n!} z^n, \qquad \sum_{n=0}^{+\infty} \frac{9i}{n^2} z^{2n}.$$

1.3. Cauchy-Riemann and Holomorphicity Show that $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = f(x+iy) = \sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin, but that it is *not* holomorphic in zero.

1.4. Geometric transformations of the complex plane Let $f : \mathbb{C} \to \mathbb{C}$ be the holomorphic function defined by $f(z) = az + b$, for some coefficients $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. Suppose that $w \in \mathbb{C}$ is a fixed point of f, that is $f(w) = w$.

(a) Show that $f(z) = a(z-w) + w$.

(b) Identifying $\mathbb{C}$ with $\mathbb{R}^2$ describe $f : \mathbb{C} \to \mathbb{C}$ as combination of geometric transformations of the plane (rotations, translations, and dilations).

1.5. $\star$ **Harmonicity** A real C^2-function $w = w(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* if its Laplacian $\Delta w = \operatorname{div}(\nabla w) := \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ is equal to zero everywhere. Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function. Denote with $u = \Re(f)$ and $v = \Im(f)$ the real part and imaginary part of f, so that $f(z) = u(z) + iv(z)$ for every $z \in \mathbb{C}$. Show that both u and v are harmonic functions by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

You can assume for now u and v of class C^2. We will see that they are in fact smooth functions.

1.6. $\star$ **Applications of CR equations** Let $\Omega \subset \mathbb{C}$ be a domain, i.e an open connected subset of $\mathbb{C}$.

(a) Let $u : \Omega \to \mathbb{R}$ be a differentiable function such that $\frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0$ for all $z \in \Omega$. Prove that u is constant on Ω.

(b) Let $f : \Omega \to \mathbb{C}$ be holomorphic and $f'(z) = 0$ for all $z \in \Omega$. Prove that f is constant in Ω.

(c) If $f = u + iv$ is holomorphic on Ω and if any of the functions u, v or $|f|$ is constant on Ω then f is constant.