Exercises with a $\star$ are eligible for bonus points.

2.1. Complex numbers and geometry I Denote with $A_y := \{iy : y \in \mathbb{R}\} \subset \mathbb{C}$ the y-axis in the complex plane. Describe geometrically the image of A_y under the exponential map $\{e^z : z \in A_y\}$. Repeat the same replacing A_y with the x-axis $A_x := \{x : x \in \mathbb{R}\} \subset \mathbb{C}$, the diagonal $D := \{a + ia : a \in \mathbb{R}\} \subset \mathbb{C}$, and the curve $\{\log(a) + ia : a > 0\} \subset \mathbb{C}$.

2.2. Complex numbers and geometry II A Möbius transformation is a map $f : \mathbb{C} \to \mathbb{C}$ defined as

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - cb \neq 0$.

(a) Show that the set of Möbius transformations form a group when endowed with the operation of composition $((f_1 \circ f_2)(z) := f_1(f_2(z)))$.

(b) Show that the image of any circle by a Möbius transformation is either a circle or an affine line.

2.3. $\star$ **Integrating over a triangle** Let Ω be an open subset of $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic, and that $f' : \Omega \to \mathbb{C}$ is continuous. Show taking advantage of the Green formula [1] that

$$\int_T f \, dz = 0,$$

where the integration is along an arbitrary triangle T contained in Ω.

2.4. Line integral I Compute the following complex line integrals. Here $\Re(z)$ and $\Im(z)$ denote respectively the real and imaginary parts of z.

(a) $\int_\gamma (z^2 + z) \, dz$, when γ is the segment joining 1 to $1 + i$.

(b) $\int_\gamma (\Re(z^2) - \Im(z)) \, dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

(c) $\int_\gamma \bar{z} \, dz$, when γ is the boundary of the half circle $\{z \in \mathbb{C} : |z| < 1, \Im(z) \geq 0\}$.

[1] Let C be a positively oriented, piecewise-smooth simple curve in the plane, and let D be the region bounded by C. If $\vec{F} = (F^1, F^2) : \bar{D} \to \mathbb{R}^2$ is a vector field whose components have continuous partial derivatives, then Green's theorem states: $\int_C \vec{F} \cdot dr = \iint_D (\partial_x F^2 - \partial_y F^1) \, dx \, dy$.

(d) Let $a, b \in \mathbb{C}$ be such that $|a| < 1 < |b|$. Denote with $C = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle in the complex plane. Show that

$$\int_C \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}.$$

2.5. ⋆ **Line integral II** Let $f : \mathbb{C} \to \mathbb{C}$ be any complex polynomial, that is $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ for some $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{C}$. Show that the line integral of f along any circle is equal to zero.

Hint: first prove this for the unit circle $\{z : |z| = 1\}$ *and* $f(z) = z^n$ *for* $n \geq 0$. *Then, deduce the general result.*