Exercises with a $\star$ are eligible for bonus points.

3.1. Complex line integrals

(a) Compute $\int_\gamma \cos(\Re(z))\,dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

(b) Compute $\int_\gamma (\bar{z})^k\,dz$ for any $k \in \mathbb{Z}$ and when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

(c) Compute $\int_\gamma (z^{2023} + \pi z^{11} + i)\,dz$, when γ is the spiral $\{1 + te^{i\pi t} : t \in [0,1]\}$.

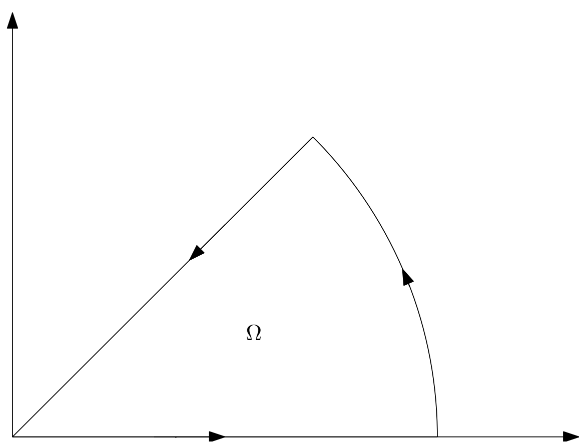
3.2. $\star$ **A polynomial identity** Let γ be the counter-clockwise oriented circle of radius $r > 0$ and center $z_0 \in \mathbb{C}$, and let p be any complex polynomial. Show that

$$\int_\gamma p(\bar{z})\,dz = 2\pi i r^2 p'(\bar{z}_0).$$

3.3. $\star$ **Real integrals via complex integration** For the first point you can use Cauchy Theorem (f holomorphic and γ closed implies $\int_\gamma f\,dz = 0$). Also, it could be useful to recall the Gaussian integral $\int_{-\infty}^{+\infty} e^{-t^2}\,dt = \sqrt{\pi}$.

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be the holomorphic function defined by $f(z) = e^{iz^2}$ and $R > 0$. By integrating f over the boundary of $\Omega = \{re^{i\theta} : r \in (0,R), \theta \in (0,\pi/4)\}$, deduce the value of the *Fresnel integrals*

$$\int_0^{+\infty} \cos(x^2)\,dx, \quad \int_0^{+\infty} \sin(x^2)\,dx.$$



(b) Let γ be the counter clockwise oriented unit circle and $n \in \mathbb{N}$. Compute

$$\int_\gamma z^{-1}(z+z^{-1})^{2n}\,dz,$$

and deduce that

$$\int_0^{2\pi} \cos(t)^{2n}\,dt = \frac{1}{2^{2n-1}}\binom{2n}{n}\pi.$$

3.4. (Challenging and optional) Approximation by polygonal curves Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$ and suppose that there exists a sequence $\gamma_n : [a,b] \to \mathbb{C}$ of polygonal curves, i.e. curves that are piecewise affine, such that $\gamma_n \to \gamma$ and $\gamma_n' \to \gamma'$ uniformly as $n \to +\infty$ (that is $\gamma_n \to \gamma$ with respect to the usual C^1-topology).

(a) Show that any closed C^2-curve γ admit such approximation.

(b) Show taking advantage of Goursat Theorem that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, f' is continuous, and γ is like in the statement of the exercise, then

$$\int_\gamma f\,dz = 0.$$