Exercises with a $\star$ are eligible for bonus points.

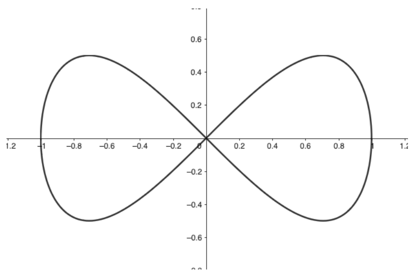
4.1. $\star$ **Cauchy Formula** Compute the following integrals

(a) $\int_0^{2\pi} e^{e^{it}}\, dt$.

(b) $\int_\gamma \frac{iz}{(z-\alpha)(z+i\alpha)}\, dz$, when $\gamma = \{z \in \mathbb{C} : |z| = 2\}$ and $\alpha \in [-1, 1]$.

(c) $\int_\gamma \frac{z^2+z}{z^2+1}\, dz$, when $\gamma = \{z \in \mathbb{C} : |z| = 3\}$.

(d) $\int_\gamma \sin(z)^2 \cos(z)\, dz$, when γ is the 'infinity symbol' $t \mapsto \sin(t) + i\sin(t)\cos(t)$, $t \in [0, 2\pi]$.



4.2. The complex logarithm Let

$$U = \mathbb{C} \setminus \{z \in \mathbb{C} : \Im(z) = 0, \Re(z) \leq 0\}$$

be the open set obtained by removing the negative real axis from the complex plane $\mathbb{C}$. The complex logarithm is defined in U as

$$\log(z) := \log(|z|) + i\arg(z), \quad z = |z|e^{i\arg(z)},$$

where $\arg(z) \in]-\pi, \pi[$. Show that for every $z \in U$

$$\log(z) = \int_\gamma \frac{1}{w}\, dw,$$

where γ is the segment connecting 1 to z.

Hint: integrate over a well chosen closed curve containing γ and passing through $|z|$.

4.3. Quotients and integration Let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be an holomorphic function, and $\gamma : [a, b] \to \Omega$ a smooth, closed curve. Suppose $|f(z) - 1| < 1$ for all $z \in \Omega$. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0.$$

4.4. $\star$ **Generalized Liouville** Prove the following theorem: every holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that

$$|f(w)| \leq c|w|^n, \quad \text{for all } w \in \{z \in \mathbb{C} : |z| > C\}$$

for some $c, C > 0$ and $n \geq 0$, is a complex polynomial of degree at most n.