Exercises with a $\star$ are eligible for bonus points.

5.1. Discrete maps A subset $\mathcal{A}$ of an domain $\Omega \subset \mathbb{C}$ is called *discrete* in Ω if it has no limit point in Ω. A function $f : \Omega \to \mathbb{C}$ is called *discrete* if for every $w \in \mathbb{C}$ the set

$$E_w := \{z \in \Omega : f(z) = w\}$$

is discrete in Ω.

(a) Let Ω be connected and open. Show that every non-constant holomorphic function $f : \Omega \to \mathbb{C}$ is discrete.

(b) Show that if Ω is compact, then $\mathcal{A} \subset \Omega$ is discrete if and only if it has finite cardinality. Is this true if Ω is merely bounded?

5.2. Order of zeros

(a) Find the zeros of the function $z \mapsto \cos(z^2)$ and determine their order.

(b) Let $f, g : \mathbb{C} \to \mathbb{C}$ two holomorphic functions that vanish simultaneously at some point $z_0 \in \mathbb{C}$ with order $a \in \mathbb{N}$ and $b \in \mathbb{N}$ respectively. Show that the function $h = f + g$ vanish at z_0 with order $c \geq \min\{a, b\}$. Give an explicit example realizing the strict inequality.

5.3. Taylor series Compute the radius of convergence of the Taylor serie of the function $f(z) = \frac{\sin(z)}{z^2 - i}$ in $z_0 = 0$ and $z_0 = 1$.

5.4. $\star$ **A complex ODE** Take advantage of the power series expansion to find $f : \mathbb{C} \to \mathbb{C}$ holomorphic such that $f'(z) = z^2 f(z)$ and $f(0) = 1$.

5.5. $\star$ **Riemann continuation Theorem** Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be holomorphic. Show that the following are equivalent:

1. There exists $g : \mathbb{C} \to \mathbb{C}$ holomorphic, such that $g(z) = f(z)$ for all $z \neq 0$.
2. There exists $g : \mathbb{C} \to \mathbb{C}$ continuous, such that $g(z) = f(z)$ for all $z \neq 0$.
3. There exists $\varepsilon > 0$ such that f is bounded in $\dot{B}_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\} \setminus \{0\}$.
4. $\lim_{z \to 0} z f(z) = 0$.

Hint: to prove 4. $\Rightarrow$ 1. *define* $h(z) = z f(z)$ *when* $z \neq 0$ *and* $h(0) = 0$. *Analyse the relation between* $f(z)$, $h(z)$ *and* $k(z) := z h(z)$.