Exercises with a $\star$ are eligible for bonus points.

6.1. Uniform convergence Let Ω be an open subset of $\mathbb{R}^M$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C^0(\Omega,\mathbb{R}^N)$. Show that if f_n converges locally uniformly[1] to some map $f:\Omega\to\mathbb{R}^N$, then $f\in C^0(\Omega,\mathbb{R}^N)$.

6.2. A generalization In Exercise 3.2 we showed the following identity holding for all polynomial p an circle $\gamma=\{z\in\mathbb{C}:|z-z_0|=r\}$: $\int_\gamma p(\bar z)\,dz = 2\pi i r^2 p'(\bar z_0)$. Extend this result to all entire functions $f:\mathbb{C}\to\mathbb{C}$, proving that

$$\int_\gamma f(\bar z)\,dz = 2\pi i r^2 f'(\bar z_0)$$

for all circle γ of radius $r>0$ centered in z_0.

6.3. $\star$ **Schwarz reflection principle** Let Ω be open, **connected**, and symmetric with respect to the x-axis (i.e. $z\mapsto\bar z$ preserves Ω), and let $f:\Omega\to\mathbb{C}$ be holomorphic. Suppose that $L:=\{z\in\Omega:\Im(z)=0\}$ is non-empty. Prove that $f(\bar z)=\overline{f(z)}$ for all $z\in\Omega$ if and only if f is real valued on L.

Hint: consider g to be the restriction of f to the upper half plane intersected with Ω. 'Reflect' g by imposing $g^*(z):=\overline{g(\bar z)}$. Argue taking advantage of Morera's Theorem.

6.4. Dense image Show that the image of an non-constant holomorphic function $f:\mathbb{C}\to\mathbb{C}$ is *dense* in $\mathbb{C}$, that is: for every $z\in\mathbb{C}$ and $\varepsilon>0$, there exists $w\in\mathbb{C}$ such that $|z-f(w)|<\varepsilon$.

Remark: The little Picard Theorem asserts in fact that $f(\mathbb{C})$ misses at most one single point of $\mathbb{C}$!

6.5. $\star$ **Geometric identity** Let $\Omega\subset\mathbb{C}$ be bounded and open with C^1-boundary. Show that

$$\int_{\partial\Omega}\bar z\,dz = 2iA(\Omega),$$

where $A(\Omega)$ denotes the measure of the area of the set Ω.

[1] Recall that local uniform convergence means that for every $x\in\Omega$ there exists an open neighbourhood U of x in Ω so that $f_n\to f$ uniformly in U: $\forall\varepsilon>0$ there exists $N\in\mathbb{N}$, such that for all $n\geq N\Rightarrow|f_n(y)-f(y)|<\varepsilon$ for all $y\in U$.