Exercises with a $\star$ are eligible for bonus points.

7.1. $\star$ **Calculus of residues** Determine the order of the poles of the following functions and compute their residue at the indicated points:

$$\operatorname{res}_{2i}\Big(\frac{1}{z^2+4}\Big), \quad \operatorname{res}_0\Big(\frac{\sin(z)}{z^2}\Big), \quad \operatorname{res}_0\Big(\frac{\cos(z)}{z^2}\Big), \quad \operatorname{res}_1\Big(\frac{1}{z^5-1}\Big).$$

7.2. Complex integrals Compute the following complex integrals taking advantage of the Residue Theorem[1].

(a)

$$\int_{|z|=2} \frac{e^z}{z^2(z-1)}\,dz.$$

(b)

$$\int_{|z|=1} \frac{1}{z^2(z^2-4)e^z}\,dz.$$

(c)

$$\int_{|z|=1/2} \frac{1}{z\sin(1/z)}\,dz.$$

(d)

$$\int_{\gamma} \frac{1}{(z-i)(z+2)(z-4)}\,dz,$$

for any simple closed curve γ that does not intersect the points $\{i,-2,4\}$.

7.3. $\star$ **Poles at infinity** Let $f:\mathbb{C}\to\mathbb{C}$ be holomorphic. We say that f *has a pole at infinity of order* $N\in\mathbb{N}$ if the function $g(z) := f(1/z)$ has a pole of order N at the origin in the usual sense. Prove that if $f:\mathbb{C}\to\mathbb{C}$ has a pole of order $N\in\mathbb{N}$ at infinity, then it has to be a polynomial of degree $N\in\mathbb{N}$.

[1] Recall: $\{z_1,\dots,z_N\}\subset\Omega$ poles and $f:\Omega\setminus\{z_1,\dots,z_N\}\to\mathbb{C}$ holomorphic. Then if $\{z_1,\dots,z_N\}$ are inside a simple closed curve γ in Ω, then $\int_\gamma f\,dz = 2\pi i\sum_{j=1}^N \operatorname{res}_{z_j}(f)$.

7.4. The Gamma function Let $Z_- := \{0, -1, -2, \dots\}$ the set of all non-positive integers, and define for all $\tau \in \mathbb{R}$ the set $U_\tau := \{z \in \mathbb{C} : \Re(z) > \tau, z \notin Z_-\}$, and $U := \mathbb{C} \setminus Z_-$.

(a) Show that the function defined by the complex improper Riemann integral

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} \, dt$$

is well defined for all $z \in U_1$. (Here $t^{z-1} = \exp((z-1)\log(t))$).

(b) Prove that Γ is holomorphic in U_1.

Hint: First show that the functions of the sequence $(\Gamma_n)_{n\in\mathbb{N}}$ given by truncating the integral at height n ($\Gamma_n(z) = \int_0^n e^{-t} t^{z-1} \, dt$) are holomorphic. Then, show that $\Gamma_n \to \Gamma$ uniformly in all compact subsets of U_1.

(c) Show that $\Gamma(z+1) = z\Gamma(z)$ for all $z \in U_1$.

(d) Deduce that Γ allows a unique holomorphic extension to U_0.

(e) Deduce that Γ allows a unique holomorphic extension to U.