Exercises with a $\star$ are eligible for bonus points.

8.1. Meromorphic functions For $z \in \mathbb{C}$ such that $\sin(z) \neq 0$ define the map

$$\operatorname{cotan}(z) = \frac{\cos(z)}{\sin(z)}.$$

(a) Show that cotan is meromorphic in $\mathbb{C}$, determine its poles and their residues.

(b) Let $w \in \mathbb{C} \setminus \mathbb{Z}$ and define

$$f(z) = \frac{\pi \operatorname{cotan}(\pi z)}{(z+w)^2}.$$

Show that f is meromorphic in $\mathbb{C}$, determine its poles and their residues.

(c) Compute for every integer $n \geq 1$ such that $|w| < n$ the line integral

$$\int_{\gamma_n} f \, dz,$$

where γ_n is the circle or radius $n + 1/2$ centered at the origin and positively oriented.

(d) Deduce that

$$\lim_{n \to +\infty} \sum_{k=-n}^{n} \frac{1}{(w+k)^2} = \frac{\pi^2}{\sin(\pi w)^2}.$$

8.2. Analytic continuation Let $f : \mathbb{C} \to \mathbb{C}$ be and entire function. Then, for every $w \in \mathbb{C}$ we can write

$$f(z) = \sum_{n=0}^{+\infty} a_n^w (z-w)^n,$$

for suitable coefficients $(a_n^w)_n$ in $\mathbb{C}$. Let $B \subset \mathbb{C}$ be an open ball. We suppose that for every $w \in B$ there exists $m \geq 0$ such that $a_m^w = 0$.

(a) For every $n \geq 0$ define the set

$$A(n) := \{w \in B : a_n^w = 0\}.$$

Show that there exists $m \geq 0$ such that $A(m)$ is uncountable.

(b) Deduce that f is a polynomial or degree at most m.

8.3. $\star$ **Real integrals** Compute the following real integrals taking advantage of the Residue Theorem.

(a)

$$\int_0^\pi \frac{\cos(4t)}{\sin(t)^2+1}\,dt.$$

(b)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx.$$

8.4. $\star$ **Quotient of holomorphic functions** Let f, g be two non-constant holomorphic functions on $\mathbb{C}$. Show that if $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$, then there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$.