Exercises with a $\star$ are eligible for bonus points.

9.1. Laurent Series A *Laurent series* centered at $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n = \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

where $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$. We define $\rho_0,\ \rho_I \in [0,+\infty]$ the *outer* and *inner* radius of convergence as

$$\rho_0 := \left(\limsup_{n\to+\infty} |a_n|^{1/n}\right)^{-1}, \qquad \rho_I := \limsup_{n\to+\infty} |a_{-n}|^{1/n}.$$

If $\rho_I < \rho_0$, we define the *annulus of convergence* as

$$\mathcal{A}(z_0,\rho_I,\rho_0) := \{z \in \mathbb{C} : \rho_I < |z-z_0| < \rho_0\},$$

with the convention $\mathcal{A}(z_0,\rho_I,+\infty) = \{z \in \mathbb{C} : \rho_I < |z-z_0|\}$, so that in particular $\mathcal{A}(z_0,0,+\infty) = \mathbb{C}\setminus\{z_0\}$.

(a) Show that if $\rho_0 > 0$, then the series

$$f_0(z) := \sum_{n=0}^{+\infty} a_n(z-z_0)^n, \quad z \in \mathcal{D}_0(z_0,\rho_0) := \{z \in \mathbb{C} : |z-z_0| < \rho_0\},$$

converges absolutely and uniformly on compact sets. Show that if $\rho_I < +\infty$, then the series

$$f_I(z) := \sum_{n=0}^{+\infty} a_{-n}(z-z_0)^{-n}, \quad z \in \mathcal{D}_I(z_0,\rho_I) := \{z \in \mathbb{C} : \rho_I < |z-z_0|\},$$

converges absolutely and uniformly on compact sets.

(b) Show that a Laurent series is divergent for any z satisfying $|z-z_0| > \rho_0$ or $|z-z_0| < \rho_I$.

(c) Deduce that the full Laurent series

$$f(z) := \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$$

defines an analytic function in $\mathcal{A}(z_0,\rho_I,\rho_0)$, and its coefficients are related to f by the formula

$$a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

for any $n \in \mathbb{Z}$ and $r \in (\rho_I,\rho_0)$.

9.2. $\star$ **Meromorphic functions** Recall the definition of $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

(a) Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Show that f has at most countably many poles.

(b) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be meromorphic on $\hat{\mathbb{C}}$. Show that f has at most finitely many poles.

(c) Deduce that if $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic on $\hat{\mathbb{C}}$, than it is a rational function.

9.3. Generalization of the Argument Principle Let $\Omega \subset \mathbb{C}$ open, $z_0 \in \Omega$ and $r > 0$ such that $\bar{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\} \subset \Omega$. Suppose that $f : \Omega \to \mathbb{C}$ is homolorphic and that $f(z) \neq 0$ on the circle $\partial D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$. Show that for any holomorphic function $\varphi : \Omega \to \mathbb{C}$ we have that

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f'}{f} \varphi \, dz = \sum_{w \in D(z_0,r): f(w)=0} (\mathrm{ord}_w\, f) \varphi(w).$$

9.4. $\star$ **Application of Rouché Theorem** Take advantage of the Rouché Theorem [1] to solve the following.

(a) Show that the polynomial

$$p(z) = z^4 + z^3 + 4z^2 + 1$$

has exactly 2 zeros in $\{z \in \mathbb{C} : 1 < |z| < 3\}$.

(b) For every $1 < \lambda$ consider the map

$$f_\lambda(z) := z + \lambda - e^z.$$

Show that f_λ has exactly one zero z_0 in the half plane $\Omega = \{z \in \mathbb{C} : \Re(z) < 0\}$. Show that $\Im(z_0) = 0$, that is z_0 belongs to the real axis.

[1] Recall: Let $f, g : \Omega \to \mathbb{C}$ holomorphic and γ a closed, simple curve in Ω such that its interior lies in Ω. If $|f(z)| > |g(z)|$ for all $z \in \gamma$, then f and $f + g$ have the same number of zeros in the interior of γ.