Exercises with a $\star$ are eligible for bonus points.

10.1. Laurent Series II Let $0 \le s_1 < r_1 < r_2 < s_2$, and set $U = \mathcal{A}(0, s_1, s_2)$ and $V = \mathcal{A}(0, r_1, r_2)$ (like in Exercise 9.1). Denote with γ_1 and γ_2 the circles of radius r_1 and r_2, respectively, positively oriented. Let $f : U \to \mathbb{C}$ be a general holomorphic function.

(a) Show that the functions

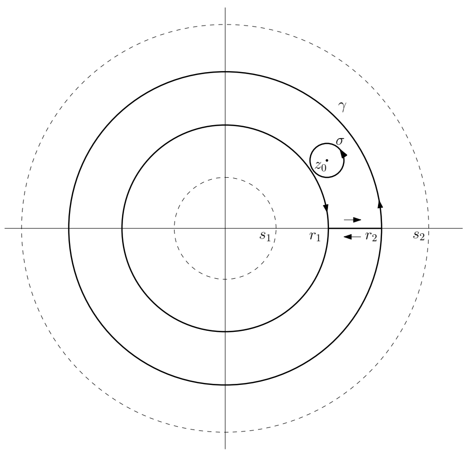
$$g_1(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z}\, dw, \quad \text{for } |z| > r_1,$$

and

$$g_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z}\, dw, \quad \text{for } |z| < r_2,$$

are well defined and holomorphic.

(b) Let γ be the closed curve obtained by going along γ_2 starting at r_2, then along the segment joining r_2 to r_1, then along $-\gamma_1$, and finally back via the segment joining r_1 to r_2. Let $z_0 \in V$ and $r > 0$ small enough such that $\sigma = \{z \in \mathbb{C} : |z - z_0| = r\}$ is



in V. Without giving a full proof, by sketching the steps of the homotopy, explain why σ and γ are homotopic in U .

(c) Show that $f = g_2 - g_1$ in V.

(d) Deduce that f can be represented as a Laurent serie, meaning: there exists a sequence $(a_n)_{n\in\mathbb{Z}}$ such that the series $\sum_{n\ge1} a_n z^n$ and $\sum_{n\ge1} a_{-n} z^{-n}$ are absolutely convergent in V, and satisfy

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \quad \text{in } V.$$

10.2. ⋆ **Logarithm** Let U be an open and simply connected domain of $\mathbb{C}$, and $f : U \to \mathbb{C}$ a non-vanishing holomorphic function. Fix $z_0 \in U$ and denote with γ_z an arbitrary curve in U connecting z_0 to z.

(a) Show that the function

$$g(z) = \int_{\gamma_z} \frac{f'}{f}\, dw,$$

is well defined and holomorphic in U, and that $g'(z) = \frac{f'(z)}{f(z)}$ for all $z \in U$.

(b) Compute the derivative of $\frac{\exp(g(z))}{f(z)}$.

(c) Deduce that there exists $\tilde{g}$ holomorphic in U such that $f = \exp(\tilde{g})$. Is this function unique?

(d) Show that for every $n \in \mathbb{N}$ there exists an holomorphic function $h_n : U \to \mathbb{C}$ such that $(h_n)^n = f$.

10.3. Complex vs Real Is it true that if $u, v : \mathbb{C} \to \mathbb{R}$ are smooth and open maps, then $f = u + iv$ is open? If true prove it otherwise give a counter example.

10.4. Symmetric Rouché

(a) Prove the following variation of Rouché's Theorem by Theodor Estermann (1962): Suppose f, g are holomorphic functions in an open domain $\Omega \subset \mathbb{C}$ and $\gamma \subset \Omega$ a simple, closed curve. If

$$|f(z) + g(z)| < |f(z)| + |g(z)|, \quad \text{for all } z \in \gamma,$$

then f and g share the same number of zeros in the interior of γ.

Hint: consider the map $tf(z) - (1-t)g(z)$.

(b) Show that the above result implies Rouché Theorem as we have seen it in class.

(c) Show with a simple counterexample that the result of point (a) is stronger than Rouché Theorem as we have seen it in class.

10.5. ⋆ **Maps preserving orthogonality** Let $\Omega \in \mathbb{R}^2$ open, and $f : \Omega \to \mathbb{R}^2$ smooth. Show that if f is orientation preserving [1] and sends curves intersecting orthogonally to curves intersecting orthogonally, then f is holomorphic (by identifying $\mathbb{R}^2$ with $\mathbb{C}$).

[1] That is the determinant of its Jacobian is positive.