

Exercises with a \star are eligible for bonus points.

12.1. Holomorphic injections Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic injection.

(a) Let $g : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ be defined as $g(z) := f(1/z)$. Show that g has no essential singularity at zero. Argue by contradiction taking advantage of the Theorem of Casorati-Weierstrass ¹.

(b) Taking advantage of the Laurent series of f and g , prove that f is in fact a polynomial.

(c) Show that f is of the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

12.2. \star Pointwise values Suppose h holomorphic in \mathbb{C} , $h(0) = 3 + 4i$, and $|h(z)| \leq 5$ if $|z| \leq 1$. What is $h'(0)$ under these conditions?

12.3. Estimates on the modulus Suppose that f is holomorphic on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$, $f(i) = 0$, and that $|f(z)| \leq 1$ for all $z \in \mathbb{H}$. How big can $|f(2i)|$ be under these conditions?

12.4. \star Schwarz-Pick's Lemma Denote with $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in \mathbb{C} , and suppose that $f : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic. Prove that for all $z \in \mathbb{D}$

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Remark: note that the above expression takes the nicer form: $|f'(a)| \leq \frac{1-|b|^2}{1-|a|^2}$, for all $a, b \in \mathbb{D}$ such that $f(a) = b$.

12.5. Symmetries of the Riemann mapping Let $\Omega \subset \mathbb{C}$ be a non-empty, simply connected domain symmetric with respect to the real axis ($\{\bar{z} : z \in \Omega\} = \Omega$). For $z_0 \in \Omega$ real, denote with $F : \Omega \rightarrow \mathbb{D}$ the unique conformal map given by the Riemann Mapping Theorem, so that $F(z_0) = 0$ and $F'(z_0) > 0$. Prove that

$$\overline{F(\bar{z})} = F(z),$$

for all $z \in \Omega$.

Hint: take advantage of Exercise 6.3. on the Schwarz reflection principle.

¹Recall: If $f : B(a, R) \setminus \{a\} \rightarrow \mathbb{C}$ holomorphic has an essential singularity in a , then for all $0 < r < R$, $f(B(a, r) \setminus \{a\})$ is dense in \mathbb{C} . (Here $B(w, \rho) := \{z \in \mathbb{C} : |z - w| < \rho\}$).