1.1. Complex Numbers Review Simplify the following expressions

$$\left(\frac{1-i\sqrt{3}}{2}\right)^{36} =$$

$$\frac{1}{i}\frac{1+2i}{1-2i} - \frac{2+4i}{1+2i} + (1+i)(1-3i) =$$

$$(1+i)^{2n}(1-i)^{2m} = \qquad \text{for every } m, n \in \mathbb{N}.$$

SOL: Notice that $\frac{1-i\sqrt{3}}{2}$ is a complex root of $z^6 = 1$. Hence

$$\left(\frac{1-i\sqrt{3}}{2}\right)^{36} = (1)^6 = 1.$$

An elementary computation shows that

$$\frac{1}{i}\frac{1+2i}{1-2i} - \frac{2+4i}{1+2i} + (1+i)(1-3i) = \frac{14-7i}{5}.$$

Since $(1+i)^2 = 2i$ and $(1-i)^2 = -2i$ we have that

$$\begin{aligned}(1+i)^{2n}(1-i)^{2m} &= 2^{m+n}(i)^n(-i)^m \\ &= -2^{m+n}(i)^{m+n} = \begin{cases} -2^{m+n}, & \text{if } m+n = 0 \bmod 4, \\ -2^{m+n}i, & \text{if } m+n = 1 \bmod 4, \\ 2^{m+n}, & \text{if } m+n = 2 \bmod 4, \\ 2^{m+n}i, & \text{if } m+n = 3 \bmod 4. \end{cases}\end{aligned}$$

1.2. Power Series Investigate the absolute convergence and radius of convergence of the following power series

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} z^n, \qquad \sum_{n=0}^{+\infty} \frac{e^{in}}{4n!} z^n, \qquad \sum_{n=0}^{+\infty} \frac{9i}{n^2} z^{2n}.$$

SOL: Let (a_n) be a sequence of complex numbers. We recall that setting

$$R = \begin{cases} \frac{1}{\limsup_{n\to+\infty}|a_n|^{1/n}}, & \text{if } \limsup_{n\to+\infty}|a_n|^{1/n} > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

the associated complex power serie $\sum_{n=0}^{+\infty} a_n z^n$ converges absolutely if $|z| < R$ and diverges if $|z| > R$. Since

$$\limsup_{n\to+\infty}\left|\frac{(-1)^n}{2n+1}\right|^{1/n} = \lim_{n\to+\infty}\frac{1}{(2n+1)^{1/n}} = 1,$$

and

$$\limsup_{n\to+\infty}\left|\frac{e^{in}}{4n!}\right|^{1/n} = \lim_{n\to+\infty}\frac{1}{(4n!)^{1/n}} = 0,$$

we have that the first power serie of the exercise is absolutely convergent if $|z| < 1$, and the second converges everywhere. Setting $k = 2n$, we can rewrite the third one as

$$\sum_{n=0}^{+\infty}\frac{9i}{n^2}z^{2n} = \sum_{k \text{ even}}\frac{36i}{k^2}z^k,$$

so that we have to check

$$\limsup_{\substack{k\to+\infty\\ k \text{ even}}}\left|\frac{36i}{k^2}\right|^{1/k} = \limsup_{k\to+\infty}\frac{36^{1/k}}{(k)^{2/k}} = 1.$$

We deduce that the radius of convergence of the third serie is equal to one.

1.3. Cauchy-Riemann and Holomorphicity Show that $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = f(x+iy) = \sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin, but that it is *not* holomorphic in zero.

SOL: In this case, setting $f = u + iv$ we have that $u(z) = u(x+iy) = \sqrt{|x||y|}$ and $v \equiv 0$. By the very definition of partial derivative, we get that

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0}\frac{0}{x} = 0,$$

and similarly $\frac{\partial u}{\partial y}(0,0) = 0$. So the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are clearly satisfied in $z = 0$. However, f is not holomorphic at the origin, since choosing for instance the particular complex increment of differentiation $H = 1 + i$, we get that

$$\lim_{\substack{\tau\to 0\\ \tau>0}}\frac{f(\tau H)-f(0)}{\tau H} = \lim_{\substack{\tau\to 0\\ \tau>0}}\frac{1}{\tau}\frac{\tau}{1+i} = \frac{1}{1+i} \neq 0,$$

On the other hand if we choose $H = 2 + i$ than a similar argument gives that the above limit is $\frac{\sqrt{2}}{2+i}$ Hence the limit of the differential quotient $(f(h) - f(0))/h$ depends on how $h \in \mathbb{C}$ approaches zero.

1.4. Geometric transformations of the complex plane Let $f : \mathbb{C} \to \mathbb{C}$ be the holomorphic function defined by $f(z) = az + b$, for some coefficients $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. Suppose that $w \in \mathbb{C}$ is a fixed point of f, that is $f(w) = w$.

(a) Show that $f(z) = a(z - w) + w$.

SOL: Since $f(w) = aw + b = w$, solving for b we get that $b = w(1 - a)$, which implies $f(z) = az + w(1 - a) = a(z - w) + w$.

(b) Identifying $\mathbb{C}$ with $\mathbb{R}^2$ describe $f : \mathbb{C} \to \mathbb{C}$ as combination of geometric transformations of the plane (rotations, translations, and dilations).

SOL: Letting $T(z) := z + w$ and $R(z) = az$ it is clear that $f = T \circ R \circ T^{-1}$, where $T^{-1}(z) = z - w$ denotes the inverse of T. The maps T and T^{-1} are translations in the w and $-w$ direction. Expressing a in polar coordinates $a = |a|e^{i\theta}$ and identifying $\mathbb{C}$ with $\mathbb{R}^2$ we get that

$$R(z) = R(x + iy) = |a|(\cos(\theta) + i\sin(\theta))(x + iy)$$

corresponds to the map

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto |a| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which is a rotation of θ radians followed by a dilation of size $|a|$. Hence, f is a rotation of the complex plane around w by $\theta = \arg(a)$ radians, followed by a dilation centered in w of size $|a|$.

1.5. $\star$ **Harmonicity** A real C^2-function $w = w(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* if its Laplacian $\Delta w = \operatorname{div}(\nabla w) := \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ is equal to zero everywhere. Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function. Denote with $u = \Re(f)$ and $v = \Im(f)$ the real part and imaginary part of f, so that $f(z) = u(z) + iv(z)$ for every $z \in \mathbb{C}$. Show that both u and v are harmonic functions by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

You can assume for now u and v of class C^2. We will see that they are in fact smooth functions.

SOL: Differentiating the first Cauchy-Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ in the y-direction, interchanging the order of differentiation and taking advantage of the second Cauchy-Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ one gets that

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial^2 y},$$

which implies $\Delta u = 0$. The same works for v by starting with the second Cauchy-Riemann equation differentiated in the y direction.

1.6. $\star$ **Applications of CR equations** Let $\Omega \subset \mathbb{C}$ be a domain, i.e an open connected subset of $\mathbb{C}$.

(a) Let $u : \Omega \to \mathbb{R}$ be a differentiable function such that $\frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0$ for all $z \in \Omega$. Prove that u is constant on Ω

SOL: Fix $z \in \Omega$ and let $w \in \Omega$ so that the segment $\gamma(t) = (1-t)z + tw$, $t \in [0,1]$, is contained if Ω. Define the function $g : [0,1] \to \mathbb{R}$ by $g(t) := u(\gamma(t))$. Since u and γ are differentiable, we have that $g \in C^1(0,1)$, with derivative

$$g'(t) = \nabla u(\gamma(t)) \cdot \gamma'(t),$$

where $\nabla u(\gamma(t)) = (\frac{\partial u}{\partial x}(\gamma(t)), \frac{\partial u}{\partial y}(\gamma(t)))$ and $\gamma'(t) = w - z$. By assumption, $\nabla u \equiv 0$ everywhere, hence $g' \equiv 0$, implying $u(w) = g(1) = g(0) = u(z)$. Suppose now $w \in \Omega$ is arbitrary. Since the domain Ω is open and connected there exists a finite sequence of points $w_0, w_1, \dots, w_N$ in Ω so that $w_0 = z$, $w_N = w$ and for every $j = 0, \dots, N-1$ the segment joining w_j to w_{j+1} is contained in Ω. Repeating the previous argument on each segment, we obtain that $u(z) = u(w_0) = u(w_1) = \dots = u(w_{N-1}) = u(w_N) = u(w)$ for all $w \in \Omega$, showing that u is indeed a constant function.

(b) Let $f : \Omega \to \mathbb{C}$ be holomorphic and $f'(z) = 0$ for all $z \in \Omega$. Prove that f is constant in Ω.

SOL: Since f is holomorphic in Ω, its real and imaginary parts are is particular differentiable in the sense of real analysis by looking at Ω as a subset of $\mathbb{R}^2$. Moreover, since $0 = f'(z) = 2\frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right)$ we have that $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \equiv 0$ in Ω. By point (a) we get that u must be constant in Ω. The same holds for v by noticing that $f'(z) = 2i\frac{\partial v}{\partial z}$. Hence, $f = u + iv$ is constant in Ω.

(c) If $f = u + iv$ is holomorphic on Ω and if any of the functions u, v or $|f|$ is constant on Ω then f is constant.

SOL: If v is constant, we get from the Cauchy-Riemann equations that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$ implying by point (a) that u is also constant, and therefore f is constant. The same holds if u is constant by interchanging the roles of u and v. If $|f| = \sqrt{u^2 + v^2}$ is constant, then also $|f|^2$ must be constant. Hence, by applying the Cauchy-Riemann identities, we get that

$$0 = \nabla |f|^2 = \left(2v\frac{\partial v}{\partial x} + 2u\frac{\partial u}{\partial x}, 2v\frac{\partial v}{\partial y} + 2u\frac{\partial u}{\partial y}\right) = \left(2v\frac{\partial v}{\partial x} + 2u\frac{\partial v}{\partial y}, 2v\frac{\partial v}{\partial y} - 2u\frac{\partial v}{\partial x}\right)$$

implying that

$$v\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial y} = 0 = v\frac{\partial v}{\partial y} - u\frac{\partial v}{\partial x}. \tag{1}$$

Now, multiplying the left hand side by v and the right hand side by u we get that the expression simplifies in

$$(u^2 + v^2)\frac{\partial v}{\partial x} = 0.$$

We have now two possibilities: if $u^2 + v^2$ vanishes somewhere, by the assumption $|f| =$ constant we get that $u = v = 0$ everywhere in Ω. Otherwise, from the above expression we deduce that $\frac{\partial v}{\partial x} = 0$ in Ω. The same argument proves that $\frac{\partial v}{\partial y} = 0$ by multiplying the left hand side of (1) by u and the right hand side by $-v$. This proves by part (a) that v is constant and hence as above f is constant.