Exercises with a $\star$ are eligible for bonus points.

2.1. Complex numbers and geometry I Denote with $A_y := \{iy : y \in \mathbb{R}\} \subset \mathbb{C}$ the y-axis in the complex plane. Describe geometrically the image of A_y under the exponential map $\{e^z : z \in A_y\}$. Repeat the same replacing A_y with the x-axis $A_x := \{x : x \in \mathbb{R}\} \subset \mathbb{C}$, the diagonal $D := \{a + ia : a \in \mathbb{R}\} \subset \mathbb{C}$, and the curve $\{\log(a) + ia : a > 0\} \subset \mathbb{C}$.

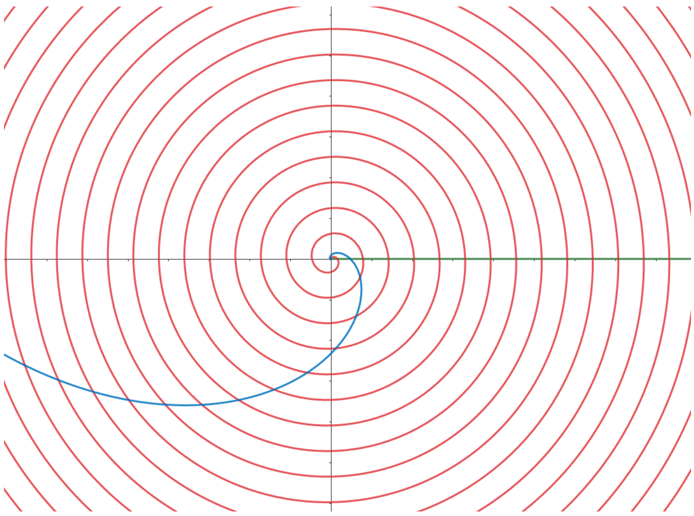
SOL: We recall that from the definiton of exponential function the following identity holds:

$$e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)),$$

for every $z = x + iy \in \mathbb{C}$. Therefore, it follows that

$$\begin{aligned}
\exp(A_y) &= \{\cos(y) + i\sin(y) : y \in \mathbb{R}\},\\
\exp(A_x) &= \{e^x : x \in \mathbb{R}\}, \text{ (in green in the picture)}\\
\exp(D) &= \{e^a(\cos(a) + i\sin(a)) : a \in \mathbb{R}\}, \text{ (in blue in the picture)}\\
\exp(G) &= \{a(\cos(a) + i\sin(a)) : a > 0\}, \text{ (in red in the picture).}
\end{aligned}$$

These sets represent geometrically in the complex plane: the unit circle, the open positive part of the x-axis, a logarithmic spiral, and an Archimedean spiral.



2.2. Complex numbers and geometry II A Möbius transformation is a map $f : \mathbb{C} \to \mathbb{C}$ defined as

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - cb \neq 0$.

(a) Show that the set of Möbius transformations form a group when endowed with the operation of composition ($(f_1 \circ f_2)(z) := f_1(f_2(z))$).

SOL: We check one by one the group axioms:

- Associativity: it follows directly from the associativity of the composition of functions.
- Identity element: the identity map $f(z) = z$ is a Möbius transformation for $(a, b, c, d) = (1, 0, 0, 0)$.
- Existence of an inverse: one can check that if $f(z) = (az + b)/(cz + d)$ is a Möbius transformation, so it is the map $g(z) = (dz - b)/(-cz + a)$, and $(f \circ g) = (g \circ f)(z) = z$.
- Closure with respect to the composition. Given two Möbius transformations $f_1(z) = (a_1 z + b_1)/(c_1 z + d_1)$ and $f_2(z) = (a_2 z + b_2)/(c_2 z + d_2)$, we have that

  $$(f_1 \circ f_2)(z) = \frac{(a_1 a_2 + b_1 c_2)z + a_1 b_2 + b_1 d_2}{(a_2 c_1 + c_2 d_1)z + b_2 c_1 + d_1 d_2} =: \frac{a'z + b'}{c'z + d'}.$$

  To check that $a'd' - c'd' \neq 0$ one can either prove it directly, or notice that

  $$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix},$$

  and take advantage of the determinant property

  $$\begin{aligned} a'd' - c'd' &= \det\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) \det\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = (a_1 d_1 - c_1 b_1)(a_2 d_2 - c_2 b_2) \end{aligned}$$

  which is different from zero because both f_1 and f_2 are Möbius transformations.

In fact, the above argument shows that the group of Möbius transformations is isomorphic to the group ${\mathrm{GL}(2,\mathbb{C})}/{\sim} = {\{A \in \mathbb{C}^{2\times 2} : \det(A) \neq 0\}}/{\sim}$ under the equivalence relation $A \sim \lambda A$, $\lambda \in \mathbb{C} \setminus \{0\}$.

(b) Show that the image of any circle by a Möbius transformation is either a circle or an affine line.

SOL: In virtue of the previous exercise, it is enough to prove this for the unit circle centered at the origin C_1. In fact, assuming the result true in this particular case, then if C is a generic circle of center z_0 and radius r and f a Möbius transformation, by setting $T(z) := rz + z_0$ one has that

$$f(C) = f(T(C_1)) = (f \circ T)(C_1),$$

and the general result follows because T is itself a Möbius transformation, and the composition preserves this property. Let $z \in C_1$ and $f(z) = (az + b)/(cz + d)$. Then reversing the relation $w = f(z)$ we get that

$$1 = |z| = |f^{-1}(w)| = \left| \frac{dw - b}{-cw + a} \right|$$

implying

$$|dw - b|^2 = |cw - a|^2 \Leftrightarrow (dw - b)(\bar{d}\bar{w} - \bar{b}) = (cw - a)(\bar{c}\bar{w} - \bar{a}),$$

which gives

$$(|d|^2 - |c|^2)|w|^2 + |b|^2 - |a|^2 - \Re(d\bar{b}w) + \Re(c\bar{a}w) = 0.$$

If $|d|^2 - |c|^2 = 0$, the above equation represents the parametric equation of an affine line (the real and imaginary parts of w appear linearly in the equation). On the other side, if $|d|^2 - |c|^2 \neq 0$, up to rescaling the coefficients, we can suppose that $|d|^2 - |c^2| = 1$. In fact, notice that if a Möbius transformation has coefficients (a, b, c, d), then the transformation with coeffients $(\lambda a, \lambda b, \lambda c, \lambda b)$ represents exactly the same map for all $\lambda \in \mathbb{C} \setminus \{0\}$. Completing the square we get that

$$0 = |w|^2 + |b|^2 - |a|^2 - \Re((d\bar{b} - c\bar{a})w) = |w - d\bar{b} + c\bar{a}|^2 - |d\bar{b} - c\bar{a}|^2 + |b|^2 - |a|^2,$$

which represents a circle with center $(d\bar{b} - c\bar{a})$ and radius $\sqrt{|d\bar{b} - c\bar{a}|^2 - |b|^2 + |a|^2}$.

2.3. Integrating over a triangle Let Ω be an open subset of $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic, and that $f' : \Omega \to \mathbb{C}$ is continuous. Show taking advantage of the Green formula [1] that

$$\int_T f \, dz = 0,$$

where the integration is along an arbitrary triangle T contained in Ω.

[1] Let C be a positively oriented, piecewise-smooth simple curve in the plane, and let D be the region bounded by C. If $\vec{F} = (F^1, F^2) : \bar{D} \to \mathbb{R}^2$ is a vector field whose components have continuous partial derivatives, then Green's theorem states: $\int_C \vec{F} \cdot dr = \iint_D (\partial_x F^2 - \partial_y F^1) \, dx \, dy$.

SOL: Write $f = u + iv$, let $\gamma(t) = x(t) + iy(t) : [a, b] \to \mathbb{C}$ be a parametrization of T, and call Ω the interior of T, that is $\partial\Omega = T$. Then, by definition of complex line integration we have that

$$\int_T f\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b (u(\gamma(t)) + iv(\gamma(t)))(x'(t) + iy'(t))\,dt$$
$$= \int_a^b (u(\gamma(t))x'(t) - v(\gamma(t))y'(t)))\,dt + i\int_a^b (u(\gamma(t))y'(t) + v(\gamma(t))x'(t))\,dt.$$

Set $\tilde{\gamma}(t) = (x(t), y(t))$ the identification of γ as a curve in $\mathbb{R}^2$. Then, defining the vector fields $\vec{F}(x, y) = (u(x, y), -v(x, y))$ and $\vec{G}(x, y) = (v(x, y), u(x, y))$ we get that the above integral is equal to

$$\int_{\tilde{\gamma}} \vec{F} \cdot dr + i\int_{\tilde{\gamma}} \vec{G} \cdot dr,$$

which by Green formula and the Cauchy-Riemann equations gives finally

$$\int_T f\,dz = \iint_\Omega (-\partial_x v - \partial_y u)\,dx\,dy + i\iint_\Omega (\partial_x u - \partial_y v)\,dx\,dy = 0.$$

2.4. Line integral I Compute the following complex line integrals. Here $\Re(z)$ and $\Im(z)$ denote respectively the real and imaginary parts of z.

(a) $\int_\gamma (z^2 + z)\,dz$, when γ is the segment joining 1 to $1 + i$.

SOL: We parametrize γ as $\gamma(t) = 1 + it$, $t \in [0, 1]$. Then

$$\int_\gamma (z^2 + z)\,dz = \int_0^1 ((1 + it)^2 + (1 + it))(1 + it)'\,dt = i\int_0^1 (2 - t^2 + 3it)\,dt$$
$$= i(2 - 1/3 + i3/2) = -3/2 + i5/3.$$

(b) $\int_\gamma (\Re(z^2) - \Im(z))\,dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

SOL: We parametrize γ as $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then

$$\int_\gamma (\Re(z^2) - \Im(z))\,dz = \int_0^{2\pi} (\cos(2t) - \sin(t))(-\sin(t) + i\cos(t))\,dt$$
$$= \int_0^{2\pi} \sin(t)^2 - \cos(2t)\sin(t)\,dt + i\int_0^{2\pi} \cos(t)\cos(2t) - \sin(t)\cos(t)\,dt$$
$$= \pi.$$

(c) $\int_\gamma \bar{z}\, dz$, when γ is the boundary of the half circle $\{z \in \mathbb{C} : |z| < 1, \Im(z) \geq 0\}$.

SOL: We parametrize the curve γ as concatenation of $t \mapsto e^{it}$ for $t \in [0, \pi]$ and $t \mapsto t$ for $t \in [-1, 1]$. Then

$$\begin{aligned}\int_\gamma \bar{z}\, dz &= \int_0^\pi (\cos(t) - i\sin(t))(-\sin(t) + i\cos(t))\, dt + \int_{-1}^1 t\, dt \\ &= i\pi.\end{aligned}$$

(d) Let $a, b \in \mathbb{C}$ be such that $|a| < 1 < |b|$. Denote with $C = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle in the complex plane. Show that

$$\int_C \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}.$$

SOL: We have that

$$\begin{aligned}\int_C \frac{dz}{(z-a)(z-b)} &= \frac{1}{a-b}\int_C \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz \\ &= \frac{i}{a-b}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-a)} - \frac{e^{it}}{(e^{it}-b)}\, dt.\end{aligned}$$

We compute this integral in two steps: first, since $|a/e^{it}| < 1$, we have that

$$\int_0^{2\pi} \frac{e^{it}}{e^{it}-a}\, dt = \int_0^{2\pi} \frac{1}{1-a/e^{it}}\, dt = \int_0^{2\pi} \sum_{n=0}^{+\infty} (-1)^n a^n e^{-itn}\, dt = 2\pi,$$

because the only term that is not zero when integrated is when $n = 0$. On the other side, since $|1/b| < 1$, we can rewrite

$$\int_0^{2\pi} \frac{e^{it}}{e^{it}-b}\, dt = \frac{1}{b}\int_0^{2\pi} \frac{e^{it}}{e^{it}/b - 1}\, dt = -\frac{1}{b}\int_0^{2\pi} e^{it}\sum_{n=0}^{+\infty} e^{int} b^{-n}\, dt = 0,$$

because term by term the integral is zero. Hence,

$$\int_C \frac{1}{(z-b)(z-c)}\, dz = \frac{4i\pi}{a-b},$$

as wished. In both cases, we took advantage of Fubini's theorem to interchange the integration with the sum. We recall the statement: let $(f_n)_{n\geq 0}$ be a sequence of functions such that

- $\int \sum_n |f_n|\, dx < +\infty$
- $\sum_n \int |f_n|\, dx < +\infty$

then $\sum_n \int f_n \, dx = \int \sum_n f \, dx$.

2.5. Line integral II Let $f : \mathbb{C} \to \mathbb{C}$ be any complex polynomial, that is $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ for some $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{C}$. Show that the line integral of f along any circle is equal to zero.

Hint: first prove this for the unit circle $\{z : |z| = 1\}$ and $f(z) = z^n$ for $n \geq 0$. Then, deduce the general result.

SOL: Denote with $C_1 = \{z : |z| = 1\}$ the unit circle. Let $n \geq 0$. Then

$$\int_{C_1} z^n \, dz = i \int_0^{2\pi} e^{int} e^{it} \, dt = \int_0^{2\pi} e^{it(n+1)} \, dt = \frac{e^{it(n+1)}}{n+1} \bigg|_0^{2\pi} = 0.$$

By linearity of the integral, we deduce that every polynomial has zero integral over the unit circle. Let C be a generic circle of radius $r > 0$ and center $z_0 \in \mathbb{C}$. Notice that $T(z) = rz + z_0$ is a complex polynomial and sends the unit circle to C. Then, we have for a generic polynomial f that

$$\int_C f \, dz = \int_{T(C_1)} f \, dz = \int_{C_1} (f \circ T) T' \, dz = r \int_{C_1} f \circ T \, dz = 0,$$

because $f \circ T$ is a polynomial, and we can apply the result previously proven.